MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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CONTENTS.

Mathematical Papers, &c.

Solved Questions.

Unsolved Questions.

No. Page

1843. (Proposed by N'IMPORTE.)—Three points being taken at random within a circle, find the chance that the circle drawn through them will lie wholly within the given circle.

1849. (Proposed by Professor SYLVESTER.)—Two points are taken at random, one on each side of a given diameter of a circle: find the probability that the chord drawn through them shall not exceed a given length.

1850. (Proposed by Professor SYLVESTER.)—In a parabolic segment cut off by a line perpendicular to the axis two points are taken; find the mean value of such of the chords drawn through them as cut the segment in two points.

1860. (Proposed by M. W. CROFTON, B.A.)—Two points being taken at random on the *perimeter* of a rectangle, find the chance of the distance between them being less than a given length. (N.B. The same law will be found to hold here as in the *Note* at the end of the solution of Quest. 132; *Reprint*, Vol. IV. p. 86.)

1869. (Proposed by W. S. BURNSIDE, B.A.)—In the analysis of the fundamental formulæ for the addition and subtraction of elliptic functions, prove geometrically that the following transformations are legitimate; viz., first change k into k^{-1}, and then change $\sin \lambda mp$ into $k \sin \lambda mp$.

1870. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Suppose there to be a promiscuous series of numbers in which the decimals have been cut off and adjusted to the nearest units of the terminal figures. Then on summing $n-1$ numbers arbitrarily taken, if from the arithmetical mean of the probabilities of the accumulated error respectively exceeding $e-\frac{1}{2}$ and $e+\frac{1}{2}$, of those units, there be subtracted $\frac{e}{n} \times$ probability of the error falling between $e-\frac{1}{2}$ and $e+\frac{1}{2}$, the difference will be the probability that the error will exceed e units when a summation includes n values.

1882. (Proposed by M. GARDINER.)—Defining the area of a curvilinear plane figure as by polar coordinates in the integral calculus, prove the following general theorem. If, at one extremity, a variable line of constant length touch in every position a plane closed re-entering curve of any form consisting of m right and of n left loops, the area of the figure described by the other extremity in the course of a complete revolution differs from that of the original figure by $(m-n)$ times the area of a circle whose radius is equal to the constant length of the line.

1884. (Proposed by FELSINIUS.)—Donner une démonstration géométrique du théorème suivant dû à M. Salmon. Si l'on a dans un plan deux systèmes de points, tels qu' à chaque point du premier système correspondent m points de l'autre, et à chaque point de ce dernier correspondent n points du premier, et si à une droite quelconque du premier système cor-

No. Page

respond dans l'autre une courbe be l'ordre l, le nombre des points doubles sera $l+m+n$.

1889. (Proposed by Chief Justice COCKLE, F.R.S.)—Solve, finitely, the differential equation $\frac{dy}{dx}+by^2=\frac{a}{x^6}$, or differential together with multiple of square of dependent variable equal to multiple of inverse sixth power of independent variable.

1891. (Proposed by Professor MANNHEIM.)—The tangents at a double point A on a quartic cut the curve again in B and C so that BC is a double tangent. Prove that $\frac{\rho_1\rho_1'}{\rho_2\rho_2'}=\frac{AB^3}{AC^3}$, where ρ_1, ρ_2 are the radii of curvature at A to the branches which touch AB, AC, and ρ_1', ρ_2' are the radii of curvature at B and C.

1907. (Proposed by W. K. CLIFFORD.)—Three ternary quadrics, U, V, W, break up into linear factors $1, 1'$; $2, 2'$; $3, 3'$, respectively. Prove that
$\square\,(U, V, W) \equiv 123.1'2'3' + 1'23.12'3' + 12'3.1'23' + 123'.1'2'3$,
where $\square\,(U, V, W)$ is the coefficient of $6\lambda\mu\nu$ in the discriminant of $\lambda U+\mu V+\nu W$, and 123 means the determinant formed with the coefficients of the linear factors 1, 2, 3. Required the developments and interpretations.

1910. (Proposed by Professor SYLVESTER.)—AB is a given straight line upon which four points are taken at random. Find the chance that their anharmonic ratio (estimated by the quotient of the whole into the middle by the product of the extreme segments) shall exceed a given quantity.

1912. (Proposed by Professor MANNHEIM.)—If a and b be the points of contact, with a curve of the third class, of a double tangent; and if this tangent be intersected in m, n, p by the three tangents to the curve which can be drawn from any point in the plane, then $\frac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp}=\frac{\rho_a}{\rho_b}$, where ρ_a, ρ_b are the radii of curvature at the points a and b.

1914. (Proposed by W. DAVIS.)—Find, to 10 decimals, *all* the roots of the equation $x^7+28x^4=480$.

1917. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Suppose the surface of a sphere to be made up of an indefinite number of points, and straight lines to be drawn through every two of those points, and determine the law of density of this mass of lines as depending on the distance from the centre of the sphere.

1918. (Proposed by W. K. CLIFFORD.)—It is known that the conic of five-pointic contact at any point A of a cubic meets the curve again in a point B constructed by joining the point A to its second tangential; let this point be called the *conic tangential* of A. Then the conic tangential of B will be the second conic tangential of A, and so on. Show how, having given the conic tangential of any order, and also the line tangential

No. Page

of any order, we can construct for the original point A by the ruler alone.

1924. (Proposed by M. W. CROFTON, B.A.)—Prove that the arc of a Cartesian oval, at any point P, is equally inclined to the straight line from P through any one focus, and to the circular arc from P through the two other foci. [The theorem in Question 1906 follows as a corollary from this.]

1933. (Proposed by C. TAYLOR, M.A.)—If α, β be the eccentric angles of two adjacent vertices of a polygon inscribed in a fixed ellipse, and if the polygon envelope a fixed confocal, then will $\Sigma \cos(\alpha-\beta) =$ a constant.

1934. (Proposed by M. W. CROFTON, B.A.)—Two bags contain m and n balls respectively; an arbitrary number is drawn from each (0 being considered a number): find the chance of the total number drawn being equal to any assigned integer, from 0 to $m+n$. [See Note to Solution of Question 1321, *Reprint*, Vol. IV., p. 86.]

1945. (Proposed by C. W. MERRIFIELD, F.R.S.)—To find a rectangular parallelepiped, such that its edges, the diagonals of its faces, and the diagonals of the solid, shall all be integral. [The Proposer states that he has not been able to solve this problem, and would therefore be glad of a solution, or of a proof that it is impossible.]

1952. (Proposed by E. PROUHET.)—Si l'on désigne par X^n_p le nombre total de manières dont un polygone convexe de n côtés peut être décomposé en p parties au moyen de $(p-1)$ diagonales qui ne se coupent pas dans l'intérieur du polygone, on a

$$X^n_p = \frac{1}{p} \cdot \frac{n(n+1)(n+2)\ldots(n+p-2)}{1.2.3\ldots(p-1)} \cdot \frac{(n-3)(n-4)\ldots(n-p-1)}{1.2.3\ldots(p-1)}.$$

Pour $n=p-2$ on retrouve la formule d'EULER démontrée dans le Journal de LIOUVILLE, tom. III. (1838), p. 505 et 547.

1956. (Proposed by R. BALL, M.A.)—If in any binary quantic $(a_0, a_1 \ldots\ldots a_n)(x, 1)^n$, or $F(x)$, x be changed into $\lambda + \dfrac{nF(\lambda)}{nx'F(\lambda)-F'(\lambda)}$, and the result be cleared of fractions by multiplying it by $\{nx'F(\lambda) - F'(\lambda)\}^n$; show that the coefficient of every power of x' in the expression thus obtained is a covariant of $F(\lambda)$.

1962. (Proposed by W. K. CLIFFORD.)—Required the *characteristics* of the system of conics having five-pointic contact with a curve of order m and class n.

1966. (Proposed by S. BILLS.)—Can the expression

$$(p^2+q^2)^4 + 64p^2q^2(p^2-q^2)^2$$

ever be a square (p and q being both rational), except when $p=q$?

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

Note on the Formula for Quadratures.

By W. S. B. Woolhouse, F.R.A.S.

The June Number of the *Educational Times* contains an article "On Approximation to a Curvilinear Area," by Professor De Morgan, in which he gives the results of a determination of the numerical coefficients appertaining to the formula usually employed in calculating quadratures. The coefficients, according to a method founded on the "Calculus of Operations," are those of the symbolic expansion of $\left\{\log(1+\Delta)\right\}^{-1}$. They may therefore be found to any number of terms by working out, by long division, the reciprocal of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ &c. Or, if C_n denote the nth coefficient, it may be computed, from those which precede it, by the formula

$$C_n = \frac{C_{n-1}}{2} - \frac{C_{n-2}}{3} + \frac{C_{n-3}}{4} - \ldots \pm \frac{C_1(=1)}{n}.$$

In allusion to these arithmetical values, Professor De Morgan's interesting Note concludes as follows:—

"The coefficients of $\left\{\log(1+\Delta)\right\}^{-1}$, so far as usually given, are

$$1,\ \tfrac{1}{2},\ -\tfrac{1}{12},\ \tfrac{1}{24},\ -\tfrac{19}{720},\ \tfrac{3}{160},\ -\tfrac{863}{60480}.$$

The three next, whether ever before printed, I know not, are

$$\tfrac{275}{24192},\ -\tfrac{33953}{3628800},\ \tfrac{8183}{1036800}."$$

The object of the present communication is merely to point out that, by an investigation conducted on the principles of ordinary Algebra, the whole of these coefficients had already been determined in my paper on "Summation," printed in the *Journal of the Institute of Actuaries*. See Vol. XI., p. 309, where, with respect to a series of equidistant ordinates V_0, V_1, V_2, &c., it is found that "the curvilinear area bounded by V_0 and V_n =

$$(V_0+V_1\ldots\ldots+V_n)-\tfrac{1}{2}(V_n+V_0)-\tfrac{1}{12}(a'-a)-\tfrac{1}{24}(b'+b)$$
$$-\tfrac{19}{720}(c'-c)-\tfrac{3}{160}(d'+d)-\tfrac{863}{60480}(e'-e)-\tfrac{275}{24192}(f'+f)$$
$$-\tfrac{33953}{3628800}(g'-g)-\tfrac{8183}{1036800}(h'+h)\ \ldots\ldots\ldots\ldots\ (B)."$$

It will be perceived that this formula includes all the coefficients stated by Professor DE MORGAN. Moreover, the general formula (A), expressing the result of the summation of an interpolated finite series of values, is carried out to the same order of differences, that is, to the eighth order. It is, however, to the integration formula (B) only that reference is needed on this occasion, and the practical inference to be drawn is, that the identity of Professor DE MORGAN's numerical coefficients with those previously determined by a process so entirely different may be accepted as a satisfactory proof of their accuracy.

1808. (Proposed by A. G.)—Décomposer un nombre triangulaire en d'autres nombres triangulaires, dans toutes les manières possibles.

Solution by SAMUEL BILLS.

Let a be the root of a given triangular number, and let x and y be the roots of two triangular numbers into which it is to be decomposed.

By the nature of triangular numbers, we must have

$$x^2+x+y^2+y = a^2+a. \quad \text{Assume} \quad y = a-\frac{r}{s}x, \text{ then}$$

$$x^2+x+a^2-2a\frac{r}{s}x+\frac{r^2}{s^2}x^2+a-\frac{r}{s}x = a^2+a;$$

therefore $$x = \frac{s(2ra+r-s)}{r^2+s^2};$$

where r and s may be any numbers that will make x and y positive integers. Let $a=6$, and take $r=1$, and $s=3$; then $x=3$ and $y=5$; thus the triangular number 21 is decomposed into two triangular numbers 6 and 15. If $a=8$, $r=1$, and $s=2$; therefore $x=6$ and $y=5$.

We might now proceed to decompose these latter numbers, where possible, as, for instance, 6 into 3 and 3. We should thus find *all* the triangular numbers of which the given one is composed. In precisely the same manner we may decompose pentagonal numbers into other pentagonal numbers, or, indeed, any m-gonal number into other m-gonal numbers.

1931. (Proposed by Professor CAYLEY.)—Find the stationary tangents (or tangents at the inflexions) of the nodal cubic

$$x(y-z)^2+y(z-x)^2+z(x-y)^2 = 0.$$

Solution by the PROPOSER.

The equation may be transformed into the form

$$(-8x+y+z)^{\frac{1}{3}}+(x-8y+z)^{\frac{1}{3}}+(x+y-8z)^{\frac{1}{3}} = 0,$$

and it thence follows immediately that the stationary tangents are the lines

$$-8x+y+z=0, \quad x-8y+z=0, \quad x+y-8z=0,$$

respectively, and that the three points of contact, or inflexions, are the intersections of these lines with the line $x+y+z=0$.

In fact, writing

$$X = kx+y+z,\quad Y = x+ky+z,\quad Z = x+y+kz,$$

we have identically

$$\begin{aligned}
&(X+Y+Z)^3-27XYZ\\
&=(k+2)^3\,(x+y+z)^3-27\,(kx+y+z)\,(x+ky+z)\,(x+y+kz)\\
&=(x^3+y^3+z^3)\,\left\{(k+2)^3-27k\right\}\\
&+3\,(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)\left\{(k+2)^3-9\,(k^2+k+1)\right\}\\
&+\qquad 3xyz\,\left\{2\,(k+2)^4-9\,(k^3+3k+2)\right\}\\
&=(k-1)^2\,(k+8)\,(x^3+y^3+z^3)+3\,(k-1)^3\,(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)\\
&\qquad -3\,(k-1)^2\,(7k+2)\,xyz.
\end{aligned}$$

Hence, writing $k=-8$, we have

$$\begin{aligned}
(X+Y+Z)^3-27XYZ &= -2187\left\{yz^2+y^2z+zx^2+z^2x+xy^2+x^2y-6xyz\right\}\\
&= -2187\left\{x\,(y-z)^2+y\,(z-x)^2+z\,(x-y)^2\right\}.
\end{aligned}$$

The equation of the given curve is therefore

$$(X+Y+Z)^3-27XYZ = 0, \text{ or } X^{\frac{1}{3}}+Y^{\frac{1}{3}}+Z^{\frac{1}{3}} = 0,$$

where of course X, Y, Z have the values

$$X = -8x+y+z,\quad Y = x-8y+z,\quad Z = x+y-8z.$$

1932. (Proposed by E. Prouhet.)—Démontrer la formule

$$\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Solution by the Proposer.

1. Soit d'abord $\phi(x)$ une fonction paire de x, c'est-à-dire une fonction telle que $\phi(x)=\phi(-x)$. Si l'on pose $z=\frac{1}{x}$, on aura

$$\begin{aligned}
\int_0^{\infty}\phi\left(x-\frac{1}{x}\right)dx &= \int_{\infty}^{0}\phi\left(\frac{1}{z}-z\right)d\,\frac{1}{z}\\
&= \int_{\infty}^{0}\phi\left(\frac{1}{z}-z\right)d\left(\frac{1}{z}-z\right)+\int_{\infty}^{0}\phi\left(\frac{1}{z}-z\right)dz.
\end{aligned}$$

Posons maintenant $\frac{1}{z}-z=u$, nous aurons pour $z=\infty$, $u=-\infty$, et $u=\infty$ pour $z=0$.

Donc $$\int_0^\infty \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty \phi(u)\,du - \int_0^\infty \phi\left(z-\frac{1}{z}\right)dz,$$

ou $$2\int_0 \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty} \phi(u)\,du.$$

Mais on a évidemment, puisque $\phi(x)$ est une fonction paire,
$$\int_{-\infty}^\infty \phi\left(x-\frac{1}{x}\right)dx = 2\int_0^\infty \phi\left(x-\frac{1}{x}\right)dx,$$
et dans le second membre de la précédente égalité on peut remplacer u par x;

donc $$\int_{-\infty}^\infty \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty \phi(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots (1).$$

ce qui démontre la proposition pour le cas d'une fonction paire.

2. Si $\psi(x)$ est une fonction impaire, on aura
$$\int_{-\infty}^\infty \psi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty \psi(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots (2),$$
car chaque membre est évidemment nul.

3. Si $f(x)$ est une fonction quelconque, on peut écrire
$$f(x) = \phi(x) + \psi(x),$$
$\phi(x)$ étant une fonction paire, et $\psi(x)$ une fonction impaire. En effet, il suffit de poser
$$\phi(x) = \tfrac{1}{2}\left\{f(x)+f(-x)\right\}, \quad \psi(x) = \tfrac{1}{2}\left\{f(x)-f(-x)\right\};$$
or si l'on ajoute les équations (1) et (2), on a
$$\int_{-\infty}^\infty \left\{\phi\left(x-\frac{1}{x}\right) + \psi\left(x-\frac{1}{x}\right)\right\}dx = \int_{-\infty}^\infty \left\{\phi(x)+\psi(x)\right\}dx.$$

ou bien $$\int_{-\infty}^\infty f\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^\infty f(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Application. $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$. Donc
$$\int_{-\infty}^\infty e^{-\left(x-\frac{1}{x}\right)^2}dx = \sqrt{\pi}, \quad \text{ou} \quad \int_{-\infty}^\infty e^{-\left(x^2+\frac{1}{x^2}\right)}dx = e^{-2}\sqrt{\pi}.$$

1936. (Proposed by the Rev. R. TOWNSEND, M.A.)—If the image of a plane conic formed by refraction through a thin lens be another plane conic, show that the cone subtending both conics from the centre of the lens will be a cone of revolution.

Solution by the PROPOSER.

The image of a plane by refraction through a thin lens being a quadric of revolution, one of whose foci is the centre of the lens; and every plane section of a quadric of revolution determining at each focus of the surface a cone of revolution; therefore, &c.

1942. (Proposed by CANTAB.)—If a, b are the semi-axes of an ellipse, and ϕ, ϕ' the eccentric angles of two points P, Q on the curve; prove that the area of the parallelogram formed by the tangents at the ends of the diameters through P and Q is $4ab \operatorname{cosec}(\phi'-\phi)$.

Solution by J. McDOWELL, M.A.; S. W. BROMFIELD; *and others.*

1. If m, n be two adjacent sides of a parallelogram, α the contained angle, and m', n' the perpendiculars on m, n, respectively, from the opposite sides, then (A) its area is $A = m'n' \operatorname{cosec}\alpha$.

For $A^2 = (mn\sin\alpha)^2 = (mm')^2 = (nn')^2 = mnm'n' = mn\sin\alpha \,.\, m'n'\operatorname{cosec}\alpha$
$= A \,.\, m'n'\operatorname{cosec}\alpha$.

2. If P, Q and P′, Q′ be corresponding points on an ellipse and its auxiliary circle, then it is known that the chords PQ, P′Q′ will meet in a point T on the major axis; and also that tangents at P and P′ will meet in the same point on the major axis; hence if the auxiliary circle be turned about the major axis of the ellipse through an angle θ such that $\cos\theta = \dfrac{b}{a}$, the orthogonal projection of the circle and of P′Q′ on the plane of the ellipse will coincide with the ellipse and PQ respectively; also the tangent at P′ will be projected into the tangent at P.

3. If A be the area of any polygon inscribed in an ellipse or circumcumscribed about it, and A′ the area of the corresponding polygon connected with the auxiliary circle, it is now evident that

$$A = A' \,.\, \frac{b}{a}.$$

4. The theorem in the question is now easily proved. For the parallelogram circumscribing the auxiliary circle at the ends of diameters corresponding to the diameters of the ellipse through P and Q, clearly has the perpendicular distances between its pairs of opposite sides each $= 2a$, and one of its angles $= \phi'-\phi$, therefore its area $= 4a^2\operatorname{cosec}(\phi'-\phi)$. Hence the area of the parallelogram formed by tangents at the extremities of P and Q is

$$\frac{b}{a} \,.\, 4a^2 \operatorname{cosec}(\phi'-\phi), \text{ or } 4ab\operatorname{cosec}(\phi'-\phi).$$

[Another proof is given by Mr. THOMSON, in the *Reprint*, Vol. III., p. 26.]

1953. (Proposed by C. W. MERRIFIELD, F.R.S.)—If three coordinate planes be inclined to one another at angles of 120°, prove that the relation between the direction-cosines of a line is

$$(\cos\lambda+\cos\mu)^2+(\cos\mu+\cos\nu)^2+(\cos\nu+\cos\lambda)^2=\tfrac{4}{3}.$$

Solution by H. TOMLINSON; S. W. BROMFIELD; REV. J. L. KITCHIN, M.A.; T. J. SANDERSON, B.A.; J. DALE; R. TUCKER, M.A.; *and many others.*

If α, β, γ be the angles between the coordinate axes, we have the known relation (see Frost's *Solid Geometry*, Art. 29),

$$\cos^2\lambda\sin^2\alpha+\cos^2\mu\sin^2\beta+\cos^2\nu\sin^2\gamma+2\cos\mu\cos\nu(\cos\beta\cos\gamma-\cos\alpha)$$
$$+2\cos\nu\cos\lambda(\cos\gamma\cos\alpha-\cos\beta)+2\cos\lambda\cos\mu(\cos\alpha\cos\beta-\cos\gamma)$$
$$=1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma\;..\;(1).$$

Now if we imagine a sphere to be drawn round the origin as centre, its intersections with the coordinate planes will form a spherical triangle whose angles are each 120° (the angles between these *planes*), and whose sides will measure the angles α, β, γ; hence we find $\cos\alpha=\cos\beta=\cos\gamma=-\tfrac{1}{3}$.

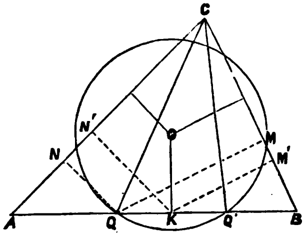
The relation (1) thus becomes, in the case proposed,

$$\cos^2\lambda+\cos^2\mu+\cos^2\nu+\cos\mu\cos\nu+\cos\nu\cos\lambda+\cos\lambda\cos\mu=\tfrac{2}{3},$$

or $$(\cos\lambda+\cos\mu)^2+(\cos\mu+\cos\nu)^2+(\cos\nu+\cos\lambda)^2=\tfrac{4}{3}.$$

1951. (Proposed by Professor WHITWORTH, M.A.)—To find the trilinear equation to the circle whose centre is at the point $(\alpha', \beta', \gamma')$ and whose radius is r.

I. *Solution by* T. J. SANDERSON, B.A.



Let ABC be the triangle of reference, O the point $(\alpha', \beta', \gamma')$ and centre of circle. The trilinear equation of the circle must be of the form

$$A\alpha^2+B\beta^2+C\gamma^2+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta=0\;......\;(1);$$

and if in this equation we make

$$\gamma=0\;........\;(2),$$

we have a quadratic in $\frac{\alpha}{\beta}$ denoting two straight lines passing through the intersections of (1) and (2), and also (since they are homogeneous) both passing through C, *i.e.* the quadratic in $\frac{\alpha}{\beta}$ is the equation of the two straight lines CQ, CQ′ in the figure.

We proceed then to form this equation. The equation to CQ is

$$\frac{\alpha}{\beta} = \frac{QM}{QN} = \frac{QK \sin B + KM'}{KN' - QK \sin A} = \frac{\sqrt{(r^2-\gamma'^2)} \sin B + \alpha' + \gamma' \cos B}{\beta' + \gamma' \cos A - \sqrt{(r^2-\gamma'^2)} \sin A},$$

or $$\alpha\left\{\beta' + \gamma' \cos A - \sqrt{(r^2-\gamma'\ )} \sin A\right\} - \beta\left\{\alpha' + \gamma' \cos B + \sqrt{(r^2-\gamma'^2)} \sin B\right\} = 0 \ldots\ldots\ldots\ldots (3).$$

Similarly the equation to CQ′ will be

$$\alpha\left\{\beta' + \gamma' \cos A + \sqrt{(r'-\gamma'^2)} \sin A\right\} - \beta\left\{\alpha' + \gamma' \cos B - \sqrt{(r^2-\gamma'^2)} \sin B\right\} = 0 \ldots\ldots\ldots\ldots (4).$$

The equation to the two lines CQ, CQ′ is, multiplying (3) and (4) together,

$$\alpha^2 (\beta'^2 + \gamma'^2 + 2\beta'\gamma' \cos A - r^2 \sin^2 A) + \beta^2 (\gamma'^2 + \alpha'^2 + 2\gamma'\alpha' \cos B - r^2 \sin^2 B)$$
$$+ 2\alpha\beta\left\{(\gamma'^2 - r^2(\sin A \sin B - (\alpha' + \gamma' \cos B)(\beta' + \gamma' \cos A)\right\} = 0.$$

This, then, is the equation which we proposed to find; and we can now write down, by symmetry, the required equation to the circle, viz.,

$$\begin{aligned}
&\alpha^2 (\beta'^2 + \gamma'^2 + 2\beta'\gamma' \cos A - r^2 \sin^2 A) \\
&+ \beta^2 (\gamma'^2 + \alpha'^2 + 2\gamma'\alpha' \cos B - r^2 \sin^2 B) \\
&+ \gamma^2 (\alpha'^2 + \beta'^2 + 2\alpha'\beta' \cos C - r^2 \sin^2 C) \\
&+ 2\beta\gamma\left\{(\alpha'^2 - r^2) \sin B \sin C - (\beta' + \alpha' \cos C)(\gamma' + \alpha' \cos B)\right\} \\
&+ 2\gamma\alpha\left\{(\beta'^2 - r^2) \sin C \sin A - (\gamma' + \beta' \cos A)(\alpha' + \beta' \cos C)\right\} \\
&+ 2\alpha\beta\left\{(\gamma'^2 - r^2) \sin A \sin B - (\alpha' + \gamma' \cos B)(\beta' + \gamma' \cos A)\right\} = 0.
\end{aligned}$$

II. *Solution by* F. D. Thomson, M.A.

The perpendicular distance from the point x', y', z' (trilinear coordinates) on the line $\lambda x + \mu y + \nu z = 0$ is

$$\frac{\lambda x' + \mu y' + \nu z'}{\sqrt{(\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C)}}$$

Hence if $x'y'z'$ be the centre of the circle, and $\lambda x + \mu y + \nu z = 0$ be a tangent, we have

$$(\lambda x' + \mu y' + \nu z')^2 = r^2 (\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C) \ldots (i).$$

Hence taking λ, μ, ν to be the *tangential* coordinates of the tangent, (i) may be regarded as the *tangential* equation to the circle.

Arranging terms, (i) becomes

$$\lambda^2 (x'^2 - r^2) + \mu^2 (y'^2 - r^2) + \nu^2 (z'^2 - r^2) + 2\mu\nu \left\{y'z' + r^2 \cos A\right\} + \&c. = 0.$$

Hence, if the corresponding *trilinear* equation be

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0, \text{ we have}$$

$$\begin{aligned}
A &= (y'^2 - r^2)(z'^2 - r^2) - (y'z' + r^2 \cos A)^2 \\
&= r^2\left\{r^2 \sin^2 A - (y'^2 + z'^2 + 2y'z' \cos A)\right\}
\end{aligned}$$

$B = \&c.$, $C = \&c.$;

$$\begin{aligned}
F &= (z'x' + r^2 \cos B)(x'y' + r^2 \cos C) - (x'^2 - r^2)(y'z' + r^2 \cos A) \\
&= r^2\left[(r^2 - x'^2) \sin B \sin C + (y' + x \cos C)(z' + x \cos B)\right]
\end{aligned}$$

$G = \&c.$, $H = \&c.$;

therefore the equation to the circle in trilinear coordinates is as given in the foregoing solution.

III. *Solution by* W. H. LAVERTY; H. TOMLINSON; *and others.*

We know that if d be the distance between two points $(\alpha\beta\gamma)$ and $(\alpha'\beta'\gamma')$ and if A, B, C, represent respectively the three determinants

$$\begin{vmatrix}\beta & \gamma\\ \beta' & \gamma'\end{vmatrix}, \quad \begin{vmatrix}\gamma & \alpha\\ \gamma' & \alpha'\end{vmatrix}, \quad \begin{vmatrix}\alpha & \beta\\ \alpha' & \beta'\end{vmatrix},$$

then

$$d = \frac{abc}{(2\Delta)^2}\left\{A^2+B^2+C^2-2(BC\cos A+CA\cos B+AB\cos C)\right\}^{\frac{1}{2}}.$$

Hence in the circle in question we have

$$\frac{r^2.(2\Delta)^4}{a^2b^2c^2} = \Sigma(\beta\gamma'-\beta'\gamma)^2-2\Sigma\left\{(\alpha\alpha'\beta'\gamma-\alpha^2\beta'\gamma'-\alpha'^2\beta\gamma-\alpha\alpha'\beta\gamma')\cos A\right\}$$

therefore $\quad \Sigma\left\{\alpha^2(\beta'^2+\gamma'^2+2\beta'\gamma'\cos A)\right\}$

$$+2\Sigma\left\{\beta\gamma(\alpha'^2\cos A-\beta'\gamma'-\alpha'\beta'\cos B-\gamma'\alpha'\cos C)\right\}$$

$$=\frac{r^2.(2\Delta)^2.(2\Delta)^2}{a^2b^2c^2}$$

$$=\frac{r^2.b^2c^2.\sin^2 A}{a^2b^2c^2}(a^2\alpha^2+b^2\beta^2+c^2\gamma^2+2bc\beta\gamma+2ca\gamma\alpha+2ab\alpha\beta)$$

$$=r^2\left\{\alpha^2\sin^2 A+\beta^2\sin^2 B+\gamma^2\sin^2 C+2\Sigma(\beta\gamma\sin B\sin C)\right\},$$

and the equation becomes

$$\Sigma\left\{\alpha^2(\beta'^2+\gamma'^2+2\beta'\gamma'\cos A-r^2\sin^2 A)\right\}$$

$$+2\Sigma\left\{\beta\gamma\left[(\alpha'^2-r^2)\sin B\sin C-(\beta'+\alpha'\cos C)(\gamma'+\alpha'\cos B)\right]\right\}=0.$$

1862. (Proposed by R. WARREN, B.A.)—Determine a system of values for (x, y, z), functions of (α, β, γ), and satisfying identically the equation $x^3+y^3+z^3-3xyz=(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma)^2$.

Solution by S. ROBERTS, M.A.; REV. J. L. KITCHIN, M.A.; *and others.*

Let θ be an imaginary cube root of unity, then the given equation may be written in the form

$$(x+y+z)(x+\theta y+\theta z)(x+\theta^2 y+\theta z)$$

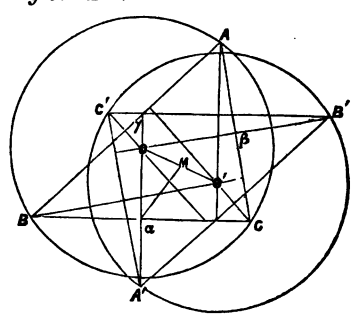
$$=\left\{(\alpha+\beta+\gamma)(\alpha+\theta\beta+\theta^2\gamma)(\alpha+\theta^2\beta+\theta\gamma)\right\}^2=k_1^2k_2^2k_3^2,$$

giving $\quad 3x=k_1^2+k_2^2+k_3^2,\quad 3y=k_1^2+\theta^2k_2^2+\theta k_3^2,\quad 3z=k_1^2+\theta k_2^2+\theta^2k_3^2.$
Having determined a set, we have six such sets by permutation.

Also if (x, y, z) satisfy the equation, $(x, \theta y, \theta^2 z)$ with their permutations also satisfy it.

1954. (Proposed by T. T. WILKINSON, F.R.A.S.)—Let ABC be any triangle; α, β, γ the middle points of BC, CA, AB, respectively; and O the centre of the circumscribing circle. Draw $O\alpha$, $O\beta$, $O\gamma$; and produce these lines to A′, B′, C′, making $OA' = 2O\alpha$, $OB' = 2O\beta$, $OC' = 2O\gamma$; and let O′ be the centre of the circle circumscribing A′B′C′. Then, if OO′ be bisected in M, the circle (M) to radius $M\alpha$ will be tangential to the *thirty-two* inscribed and escribed circles of the system of triangles ABC, A′B′C′, AO′B, BO′C, CO′A, A′OB′, B′OC′, C′OA′.

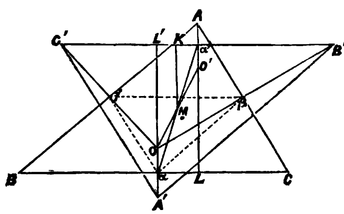
I. *Solution by* J. DALE.



Drawing the figure, it is easy to show that the triangle A′B′C′ is equal in all respects to ABC (see *Reprint*, Vol. I., pp. 6, 7), and so situated that O is the intersection of the perpendiculars of A′B′C′, while O′ is the intersection of the perpendiculars of ABC.

The circle (M) to radius ($M\alpha$) is the nine-point circle of every triangle in the system, and consequently by a known theorem tangential to the 32 inscribed and escribed circles of the 8 triangles ABC, A′B′C′, AO′B, BO′C, CO′A, A′OB′, B′OC′, C′OA′.

II. *Solution by* T. J. SANDERSON, B.A.



Since α, β, γ are the middle points of the sides, the triangle $\alpha\beta\gamma$ is similar to the triangle ABC, and of half its linear dimensions. Again, since α, β, γ are the middle points of the lines OA′, OB′, OC′, the triangle A′B′C′ is similar to the triangle $\alpha\beta\gamma$ and of *double* its linear dimensions. Hence the triangle A′B′C′ is similar and equal to the triangle ABC, but oppositely placed, homologous sides being parallel and M the centre of symmetry.

Produce A′O to meet C′B′ in L′, and A′L′ being perpendicular to BC is also perpendicular to B′C′. Draw MK perpendicular to C′B′. Then because OM = O′M, therefore $L'K = K\alpha'$ and the circle with centre M and radius $M\alpha'$ will therefore pass through L′; and it also passes through α, the middle point of OA′, therefore it is the nine-point circle of the triangle A′B′C; and similarly it is the nine-point circle of the triangle ABC. Therefore it passes through the points α, β, γ, α, β', γ', L, M, N, L′, M′, N′; (L, M, N, L′, M′, N′ are the feet of perpendiculars not all marked in the figure).

Again, because it passes through the points α', γ, and L, therefore it is the nine-point circle of the triangle AO′B, and similarly of the triangles BO′C, CO′A, A′OB′, B′OC′, C′OA′.

Now it is a known property of the nine-point circle of any triangle, that it touches the inscribed and three escribed circles of that triangle. Hence the circle with centre M and radius Mα touches the 32 inscribed and escribed circles of the 8 triangles ABC, A′B′C′, AO′B, BO′C, CO′A, A′OB′, B′OC′, C′OA′.

1955. (Proposed by C. M. INGLEBY, LL.D.)—Show that

$$\frac{a^{4b+1}-a}{30} \text{ is an integer.}$$

Solution by the REV. J. BLISSARD.

This Question, put in an extended form, may be expressed as follows:—Let $p_1, p_2 \ldots . p_n$ be prime numbers, all differing from each other; and let M be the least common multiple of the quantities $p_1-1, p_2-1, \ldots . p_n-1$. Required to prove that $\dfrac{a(a^{Mb}-1)}{p_1p_2\ldots p_n}$ is integral.

Proof.—By Fermat's Theorem, if p is a prime number, and if a does not contain p as factor, then $\dfrac{a^{p-1}-1}{p}$ is integral;

herefore $a^{p-1}-1$ is of the form mp (m integral), and $a^{p-1}=1+mp$;

$\therefore a^{(p-1)b}=(1+mp)^b$, and is of the form $1+m'p$, $\therefore \dfrac{a^{(p-1)b}-1}{p}$ is integral,

i. e., $\dfrac{a^x-1}{p}$ is integral if x contains $p-1$ as factor. Hence, generally, $\dfrac{a^x-1}{p_1p_2\ldots . p_n}$ is integral if x contains all the quantities $p_1-1, p_2-1, \ldots .$ p_n-1 as factors, *i. e.*, if $x = Mb$, provided a does not contain any of the quantities $p_1, p_2, \ldots . p_n$ as factors. Hence $\dfrac{a(a^{Mb}-1)}{p_1p_2\ldots . p_n}$ must be integral, a being any positive integer; since, if a contains any of the p numbers as factors, $a^{Mb}-1$ must contain the rest. The following are examples:—

$$\frac{a(a^b-1)}{2},\ \frac{a(a^{2b}-1)}{2.3},\ \frac{a(a^{4b}-1)}{2.3.5} \text{ (the case supposed),}$$

$$\frac{a(a^{6b}-1)}{2.3.7},\ \frac{a(a^{10b}-1)}{2.3.5.11},\ \frac{a(a^{12b}-1)}{2.3.5.7.13} \text{ are all integral.}$$

COR.—If r of the p numbers, viz., $p_1, p_2, \ldots . p_r$, are repeated, and if M is the least common multiple of $p_1-1, p_2-1, \ldots . p_r-1$; then, from above,

$$\frac{a^2(a^{Mb}-1)(a^{M'b'}-1)}{p_1^2p_2^2\ldots . p_r^2 . p_{r+1}\ldots p_n} \text{ is integral; and so on generally.}$$

II. *Solution by* Dr. Booth, F.R.S.; H. Tomlinson; Rev. J. L. Kitchin, M.A.; M. Collins, B.A.; H. Murphy; *the* Proposer; *and others.*

The expression $\dfrac{a^{4x+1}-a}{30}$ may be put under the form

$$\frac{(a^x-1)\,(a^x+1)\,(a^{2x}+1)}{30a^{x-1}}.$$

Now as all numbers are of the forms $5n$, $(5n \pm 1)$, or $(5n \pm 2)$, and as the first three factors in the numerator are manifestly divisible by 2 and 3, unless they are also divisible by 5 it will be easy to show that $a^{2x}+1$ is divisible by 5.

For the forms $5n$ and $(5n \pm 1)$ being excluded by supposition, $a^{2x}+1 = (5n \pm 2)^2 + 1 = 25n^2 \pm 20n + 5$, which is divisible therefore by 5.

Hence 2, 3 and 5 are factors of the numerator; and if they should be factors of a^x they must also be factors of a, for a^x can contain no prime factors such as 2, 3, or 5 that are not also contained in a.

Hence $a\,(a^x-1)\,(a^x+1)\,(a^{2x}+1)$ is divisible by 30 as well as

$$a^x\,(a^x-1)\,(a^x+1)\,(a^{2x}+1);$$

or $$\frac{a^{5x}-a^x}{30} \text{ and } \frac{a^{4x+1}-a}{30} \text{ are both integers.}$$

III. *Solution by* W. H. Laverty; T. J. Sanderson, B.A.; S. Bills; S. W. Bromfield; R. Tucker, M.A.; *and others.*

To show generally that $\dfrac{a^{4b+d}-a^{4c+d}}{30}$ is an integer.

We know that, whatever be the last digit in any number, that number will still end in the same digit when raised to the $(4x+1)$th power.

Now $$\frac{a^{4b+d}-a^{4c+d}}{30} = \frac{a^{d-1}}{30}\left(a^{4b+1}-a^{4c+1}\right)$$

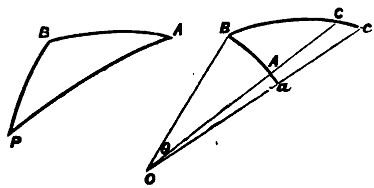
and both the numbers in the bracket will end in the same digit, therefore the difference of the two will end in a cipher and will therefore be divisible by 10. Again, a must be of one of the forms $3x-1$, $3x$, or $3x+1$; if of the second of these, the expression in the numerator is evidently divisible by 3; if of either of the others, we have

$$\frac{a^{4b+d}-a^{4c+d}}{30} = \frac{a^d}{30}\left\{(3x \pm 1)^{4b} - (3x \pm 1)^{4c}\right\}$$ and the "unities" in the expression under the bracket will cancel out, leaving the rest divisible by 3; therefore the whole expression in the numerator is divisible by 3×10, or 30.

Let now $d = 1$; $c = 0$; and we see that $\dfrac{a^{4b+1}-a}{30}$ is an integer.

1960. (Proposed by E. FITZGERALD.)—Required the curve bounding a sun-dial, whose plane is given in position, such that the length of its arc, measured from the 12 o'clock hour line, may be proportional to the time from 12 o'clock.

Solution by MATTHEW COLLINS, B.A.



Let P be the pole, $PB = a$ the meridian, $AB = \theta$ on the plane of the dial, O the point or centre where the line or style OP, *parallel to the earth's axis*, meets the plane of the dial, BC*c* the required curve, $BC = C$, $Cc = dC$, $AOB = \theta$, $AOa = d\theta$.

The spherical triangle APB gives $\cot\theta \sin a = \cot P \sin B + \cos a \cos B$, which differentiated gives $\sin a \operatorname{cosec}^2\theta \,.\, d\theta = \sin B \operatorname{cosec}^2 P \,.\, dP$. Now, by the Question, we must have $C = m \,.\, P$, therefore $m^2 (dP)^2 = (dC)^2 = (dr)^2 + (rd\theta)^2$. By eliminating P and dP from this last equation by means of our first two equations, we find a differential equation containing only dr and $d\theta$, whose integral will be the equation of the required curve.

1967. (Proposed by M. COLLINS, B.A.)—If an ellipse whose foci are G and G′ be inscribed in a plane triangle ABC, and if one focus G be the centroid of the triangle, prove (1) that the distances of the other focus G′ from the sides of the triangle are as the lengths of those sides, (2) that the sum of the squares of those distances is a minimum, and (3) that the distances of G′ from the angles are $G'A = a' \operatorname{cosec} A \div (\cot A + \cot B + \cot C)$, &c. &c., where a', b', c' are the distances from A, B, C to the middle points of BC, CA, AB.

Solution by J. DALE; T. J. SANDERSON, B.A.; REV. R. H. WRIGHT, M.A.; R. TUCKER, M.A.; *and many others.*

1. Let (x, y, z), (x', y', z') be the trilinear coordinates of G and G′; then, by well known properties,

$$xx' = yy' = zz' = (\text{semi-axis minor})^2, \quad \text{and} \quad ax = by = cz;$$

dividing each term in the first series by the corresponding term in the second, we have $x' : a = y' : b = z' : c$, which proves the theorem (1).

2. Suppose it is required to find the trilinear coordinates of a point within a triangle such that the sum of the squares of these coordinates may be a minimum. We have

$$\phi = x'^2 + y'^2 + z'^2 = \text{minimum}, \quad \text{and} \quad ax' + by' + cz' = 2\Delta;$$

differentiating, $\quad x'\,dx' + y'\,dy' + z'\,dz' = 0, \quad a\,dx' + b\,dy' + c\,dz' = 0;$

whence we find $x' : a = y' : b = z' : c$; and, as this is the point found in (1), the theorem (2) is thereby proved.

3. From the coordinates of G and G′, it appears that GA, GB, GC make the same angles with AB, AC; BC, BA; CA, CB, that G′A, G′B, G′C make with AC, AB; BA, BC; CB, CA. Hence we readily find

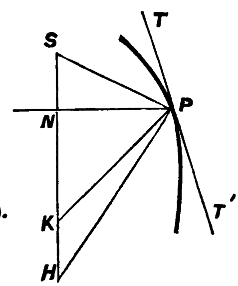
$$\frac{G'A}{a' \operatorname{cosec} A} = \frac{G'B}{b' \operatorname{cosec} B} = \frac{G'C}{c' \operatorname{cosec} C} = \frac{2x'}{a} = \frac{2y'}{b} = \frac{2z'}{c} = \frac{4\Delta}{a^2+b^2+c^2}$$

$$= \frac{2\Delta}{bc\cos A + ca \cos B + ab \cos C} = \frac{1}{\cot A + \cot B + \cot C},$$

which proves the theorem (3).

1736. (Proposed by R. Tucker, M.A.)—Any number of chords of a given fixed plane curve are drawn through a fixed point, and are traversed by a number of perfectly elastic particles, moving from rest at the fixed point; find the form of the curve, if, after impact on the perfectly smooth arc, all the particles pass through another fixed point vertically below the former.

Solution by the Proposer.



Let S, H, be the two fixed points, SP one of the chords, and TPT′ the tangent at P. Draw the horizontal line PN, and suppose the particle after impact to move initially at an $\angle$ NPK ($=\epsilon$) with PN; then if SH $= c$, NK $= z$, SN $= x$, PN $= y$, $\angle$ PSN $= \theta$, $\angle$ HPN $= \iota$, we have

$$(y \sec \iota)\cos^2\iota = 4\kappa \cos\epsilon \sin(\iota - \epsilon),$$

(by the theory of projectiles), and

$$\frac{\sin(\iota-\epsilon)}{\cos\iota} = \frac{c-x-z}{(y^2+z^2)^{\frac{1}{2}}};\ \therefore y^2+z^2 = 4x(c-x-z)\ (1).$$

Again, $\dfrac{z}{y} = \cot \text{PKN} = \cot(\theta - 2\psi)$

$$\left(\text{where} \quad \tan\psi = \frac{dy}{dx} = p\right)$$

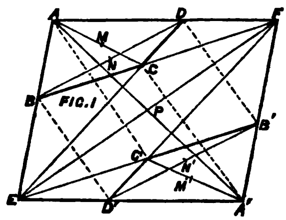
$$= \frac{\cot 2\psi \cot\theta + 1}{\cot 2\psi - \cot\theta} = \frac{x(1-p^2)+2py}{y(1-p^2)-2px} = -\frac{2x \mp (4cx-y^2)^{\frac{1}{2}}}{y}, \text{ by (1)};$$

$$\therefore p^2\left\{3xy \mp y(4cx-y^2)^{\frac{1}{2}}\right\} - 2p\left\{1 \pm x(4cx-y^2)^{\frac{1}{2}} + y^2 - 2x^2\right\}$$
$$- \left\{3xy \mp y(4cx-y^2)^{\frac{1}{2}}\right\} = 0.$$

The solution of this equation will give the curve required.

1592. (Proposed by N'IMPORTE.)—To prove that the middle points of the three diagonals of a complete quadrilateral are in the same straight line.

I. *Solution by* D. M. ANDERSON.



Let ABCDEF (Fig. 1) be a complete quadrilateral, and M, N, P, the middle points of its three diagonals AC, BD, EF. Complete the parallelogram AEA'F; draw EB', FD', intersecting in C', parallel to BF, DE, respectively, and let M', N' be the middle points of A'C', B'D'. Now the diagonals of each of the parallelograms BDB'D', ACA'C' obviously intersect in P; and therefore NN', MM', respectively parallel to their sides BD', A'C', and bisecting the aforesaid parallelograms, pass each through P. But if FD' and AE (produced) meet (say) in L, we have

$$LD' : LF = LE : LA, \text{ and } LC' : LF = LE : LB,$$

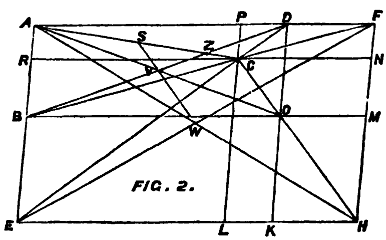
therefore $$LD' : LC' = LB : LA$$

and therefore BD' is parallel to AC'. Hence NN', MM' are parallel to each other, and passing each through one point P they must wholly coincide; that is to say, M, N, P are in one straight line.

COROLLARY.—From the above we see that if AEA'F is any parallelogram, and if EB', BF are parallel to each other, and *any* straight line FC'D' is drawn, then BD' is parallel to AC'.

[Mr. ANDERSON remarks that the parallelism of AC' and BD' may be otherwise proved, though not so briefly, by the aid of the *first* book of Euclid.]

II. *Solution by* MATTHEW COLLINS, B.A.



Let ABCDEF be the complete quadrilateral; AC, BD, EF being its three diagonals; and draw lines parallel to AE, AF, as in Fig. 2: then from the parallelograms ABMF, AEKD, we have, by Euc. I. 43, parallelogram AC = CM = CK, and therefore parallelogram NO = OL. Hence by Euc. I. 43 (*ex absurdo*), the points C, O, H are in the same straight line. Now as the diagonals of a parallelogram bisect each other, the middle points of BD, EF are the same as the middle points of AO, AH; but the middle points S, V, W of AC, AO, AH lie evidently in a straight line parallel to COH; hence the theorem is proved.

COR. 1. It follows, by projection, that if the three diagonals AC, BD, EF of a complete quadrilateral be cut by any straight line at P, Q, R, the harmonic conjugates of the points P, Q, R, taken on these diagonals, will be in the same straight line.

COR. 2. By polar reciprocation it follows that if A, B, C, D be four points and O any fifth point; and if through E (the point of intersection of AB and CD) we draw the straight line EO′, the harmonic conjugate of EO relative to EBA, ECD; and draw also the harmonic conjugates of FO and ZO, relative to the lines AD, BC (meeting in F), and to AC, BD (meeting in Z): then these three harmonic conjugates of EO, FO, ZO will meet in one point.

[The theorems in Mr. COLLINS's corollaries (the first of which includes that in the question) will furnish good exercises in the combined methods of tangential and trilinear coordinates. Putting (x, y, z) for tangential coordinates, the respective equations of the points A, B, C, E, F, D may be written

$$x = 0, y = 0, z = 0, lx + my = 0, my + nz = 0, lx + my + nz = 0 \; \ldots\ldots (1).$$

Moreover, the equations of three points (P, Q, R) and of their harmonic conjugates (P′, Q′, R′), which form on the diagonals AC, BD, EF the harmonic ranges (APCP′), (BQDQ′), (ERFR′), may be written

$$x \pm \lambda z = 0, \; (lx + my + nz) \pm \mu y = 0, \; (lx + my) \pm \nu (my + nz) = 0 \ldots\ldots (2);$$

one set of signs applying for (P, Q, R) and the other two for their harmonic conjugates (P′, Q′, R′).

Now the points (P, Q, R) or (P′, Q′, R′) will be in a straight line according as the determinant

$$\begin{vmatrix} 1, & 0\;, & \pm\lambda \\ l, & m \pm \mu\;, & n \\ l, & m \pm \nu m\;, & \pm \nu n \end{vmatrix}$$

is zero for the upper or the lower signs; but this determinant, when expanded, becomes

$$m(\lambda\nu l - n) - \mu(\lambda l - \nu n)$$

for both sets of signs: hence, if it vanish for one set of signs, it will vanish also for the other set: if, therefore (P, Q, R) be in a straight line, (P′, Q′, R′) will also be in a straight line, which proves the theorem in Mr. COLLINS's first corollary.

The same equations, with a different interpretation, will furnish a proof of the theorem in the second corollary. For considering now (x, y, z) as trilinear coordinates, equations (1) denote the sides and diagonals (EA, AD, DE, AC, BD, BF) of a complete quadrilateral ABCDEF; and equations (2) are those of the harmonic pencils (E.AODO′), (F.BOAO′), (Z.CODO′); and the same determinant shows that if the lines EO, FO, ZO meet in a point O, their harmonic conjugates will also meet in a point O′, which proves the theorem in the second corollary.]

1613. (Proposed by R. TUCKER, M.A.)—Find the locus of the vertices of a system of similar ellipses described upon the diameters of a given similar ellipse as parameters; also find the envelope of the *other* parameters.

Solution by the PROPOSER; E. MCCORMICK; E. FITZGERALD; *and others.*

Take for axes the principal axes of the fixed ellipse; then, $2a'$ being the major axis of one of the ellipses, the locus of the vertices (ρ, ϕ) will be given by

$$\rho = a'(1 \pm e), \quad r = a'(1-e^2), \quad \phi = \tfrac{1}{2}\pi \pm \theta,$$

$$r^2 = \frac{a^2(1-e^2)}{1-e^2\cos^2\theta}; \therefore \rho^2 = \frac{a^2(1 \pm e)}{1 \mp e} \cdot \frac{1}{1-e^2\sin^2\phi},$$

the equations to two ellipses.

We have for equation to the other parameter

$$Y - X\tan\theta = -2a'e\sec\theta = -\frac{2ae\sec\theta}{(1-e^2)^{\frac{1}{2}}(1-e^2\cos^2\theta)^{\frac{1}{2}}}$$

and differentiating, $X = \dfrac{2ae\sin\theta\,(1-2e^2\cos^2\theta)}{(1-e^2)^{\frac{1}{2}}(1-e^2\cos^2\theta)^{\frac{3}{2}}}$;

hence the *radial* of the envelope is found to be

$$(1-e^2)^{\frac{1}{2}}(1-e^2\sin^2\theta)^{\frac{5}{2}}r = 2ae\left\{1 + e^2 - 2e^2(2-e^2)\sin^2\theta\right\}.$$

1921. (Proposed by R. TUCKER, M.A.)—n counters are marked with the numbers 1, 3, 5.... $(2n-1)$ on both faces, and a person taking one is to have as many shillings as the number marked on the counter; find the value of a person's expectation who takes one after m have been drawn. Also find the value of the expectation when one side only is marked as above, the other sides being marked with the even numbers up to $2n$.

Solution by S. BILLS; H. HOSKINS; *and others.*

Taking the question in a more general form; suppose the n counters to be marked with the numbers $a_1, a_2, a_3 .. a_n$ on one side, and with the numbers $b_1, b_2, b_3 ... b_n$ on the other side; and let it be required to find the value of the expectation of a person drawing the n counters in succession, the numbers representing shillings.

The total amount of the numbers on the n counters being $= \Sigma(a) + \Sigma(b)$, and the number of counters being n, the value of the expectation of drawing the first will be $\dfrac{1}{n}\left\{\Sigma(a) + \Sigma(b)\right\}$ shillings.

Again; after one has been drawn, since there will $n-1$ remaining, the value of the expectation of drawing the second will be

$$\frac{1}{1-n}\left\{\Sigma(a) + \Sigma(b) - \frac{\Sigma(a) + \Sigma(b)}{n}\right\} = \frac{1}{n}\left\{\Sigma(a) + \Sigma(b)\right\}$$

shillings, the same as for the first.

It thus appears that the value of the expectation of drawing every one in succession will be the same, and equal to $\dfrac{1}{n}\left\{\Sigma(a) + \Sigma(b)\right\}$.

Now, in the first part of the question, $\Sigma(a) = \Sigma(b) = n^2$, therefore the value required $= 2n$. In the second part $\Sigma(a) = n^2$ and $\Sigma(b) = n^2 + n$, therefore the value required $= 2n + 1$.

1905. (Proposed by Chief Justice COCKLE, F.R.S.)—

If $\frac{d^2y}{dx^2}+ax^my=f(m)$, show that the solution of $f(m)=0$ may be made to depend upon that of $f\left(-\frac{m}{m+1}\right)=0$. Show also that the solution of $\frac{d^2y}{dx^2}+\phi\left(\pm\frac{1}{x}\right)\frac{1}{x^4}y=0$ may be made to depend upon that of $\frac{d^2y}{dx^2}+\phi(x)\,y=0$, the function ϕ denoting any function whatever.

Solution by the PROPOSER.

LEMMA.—If
$$\frac{d^2y}{dx^2}+ry=0 \quad\text{..........}\quad (1)$$
be soluble, then also
$$\frac{d^2y}{dx^2}+\frac{2}{x}\cdot\frac{dy}{dx}+ry=0 \quad\text{..........}\quad (2)$$
is soluble.

For if, in (1), we substitute xY for y and divide by x, the result, after replacing Y by y, will be (2).

I. Divide (1) by r, differentiate, then multiply into r, and in the product replace $\frac{dy}{dx}$ by y. The result will be
$$\frac{d^2y}{dx^2}-\frac{1}{r}\cdot\frac{dr}{dx}\cdot\frac{dy}{dx}+ry=0 \quad\text{..........}\quad (3).$$
Change the independent variable from x to t, and (3) becomes, after a slight reduction,
$$\frac{d^2y}{dt^2}-\left(\frac{1}{r}\cdot\frac{dr}{dt}+\frac{dt}{dx}\cdot\frac{d^2x}{dt^2}\right)\frac{dy}{dt}+r\left(\frac{dx}{dt}\right)^2y=0 \quad\text{..........}\quad (4).$$
Let
$$-\frac{1}{r}\cdot\frac{dr}{dt}-\frac{dt}{dx}\cdot\frac{d^2x}{dt^2}=\frac{c}{t} \quad\text{..........}\quad (5),$$
then, integrating and reducing, and denoting by G the arbitrary constant,
$$r\frac{dx}{dt}=\mathrm{G}t^{-c} \quad\text{..........}\quad (6).$$
Next, put $r=ax^m$, and integrate again; we may say, making the second arbitrary constant zero,
$$\frac{ax^{m+1}}{m+1}=\frac{\mathrm{G}t^{1-c}}{1-c} \quad\text{..........}\quad (7),$$
whence
$$x=\left(\frac{1+m}{1-c}\cdot\frac{\mathrm{G}}{a}\right)^{\frac{1}{m+1}}\cdot t^{\frac{1-c}{1+m}},\quad\text{and}\quad \frac{dx}{dt}=\left(\frac{\mathrm{G}}{a}\right)^{\frac{1}{m+1}}\cdot\left(\frac{1-m}{1-c}\right)^{\frac{-m}{m+1}}\cdot t^{-\frac{c+m}{1+m}}$$
$$\text{..........}\quad (8, 9),$$
also
$$r\left(\frac{dx}{dt}\right)^2=ax^m\left(\frac{dx}{dt}\right)^2=a^{-\frac{1}{m+1}}\cdot\mathrm{G}^{\frac{m+2}{m+1}}\left(\frac{1+m}{1-c}\right)^{-\frac{m}{m+1}}\cdot t^{-\frac{(m+2)c+m}{m+1}} \quad (10),$$
and if we make
$$\mathrm{G}=a\left(\frac{1+m}{1-c}\right)^{\frac{m}{m+2}},\quad k=-\frac{(m+2)c+m}{m+1} \quad\text{......}\quad (11, 12),$$

we shall have made the solution of $f(m) = 0$ depend upon that of

$$\frac{d^2y}{dt^2} + \frac{c}{t} \cdot \frac{dy}{dt} + at^k = 0 \ldots\ldots\ldots\ldots (13),$$

in which I here suppose that c is not equal to unity.

In (11, 12) and (13), make $c=0$, then $k = -\frac{m}{m+1}$, and if in the reduced (13) we replace t by x, that equation becomes

$$\frac{d^2y}{dx^2} + ax^{-\frac{m}{m+1}} = 0 \ldots\ldots\ldots\ldots (14).$$

In other words, we have made the solution of $f(m) = 0$ depend upon that of $f\left(-\frac{m}{m+1}\right) = 0$. The cases of $m = -1$ and $m = -2$ are exceptional.

SCHOLIUM.—The Lemma shows that connected solutions will be obtained when $c=2$, in which case $k = -\frac{3m+4}{m+1}$. Hence the solution of $f(m) = 0$ may also be made to depend upon that of $f\left(-\frac{3m+4}{m+1}\right) = 0$. Further, if the solutions of $f(m) = 0$ and $f\{\psi(m)\} = 0$, ψ being a functional symbol, are connected, then also the solutions of

$$f(m) = 0, \quad f\{\psi(m)\} = 0, \quad f\{\psi^2(m)\} = 0, \quad \ldots\ldots \quad f\{\psi^r(m)\} = 0, \text{ \&c.}$$

are connected, one with every other. Again, χ being another functional symbol, and $f(m) = 0$ and $f\{\chi(m)\} = 0$ being connected in solution, every equation of the system represented by

$$f\{\psi^r[\chi^q(m)]\} = 0 \ldots\ldots\ldots\ldots (15),$$

wherein r and q may receive any integral values whatever, is connected with every other. And the same holds when, instead of two functional symbols ψ and χ, we have any number of such symbols with any indices. In (15) let $r=q=1$, $\chi(m) = -\frac{m}{m+1}$, and $\psi(m) = -\frac{3m+4}{m+1}$,

then $$f\{\psi[\chi(m)]\} = f(-m-4) = 0 \ldots\ldots\ldots\ldots (16)$$

is connected with $f(m) = 0$. This is a particular case of the theorem which I am about to demonstrate.

II. In virtue of the Lemma the solution of $\frac{d^2y}{dx^2} + \phi(x)y = 0 \quad \ldots\ldots (17)$

may be made to depend upon that of $\frac{d^2y}{dx^2} + \frac{2}{x} \cdot \frac{dy}{dx} + \phi(x)y = 0 \ldots\ldots (18)$.

Let $$x = -\frac{1}{t}, \quad \therefore \frac{dx}{dt} = \frac{1}{t^2}, \quad \therefore \frac{d^2x}{dt^2} = -\frac{2}{t^3} \ldots\ldots\ldots (19),$$

and change the independent variable from x to t. Then (18) becomes, after slight reductions, successively

$$\frac{d^2y}{dt^2} + \left(\frac{2}{x} \cdot \frac{dx}{dt} - \frac{dt}{dx} \cdot \frac{d^2x}{dt^2}\right) \frac{dy}{dt} + \phi(x) \left(\frac{dx}{dt}\right)^2 y = 0 \ldots\ldots (20)$$

and $$\frac{d^2y}{dt^2} + \left(-\frac{2}{t} + \frac{2}{t}\right)\frac{dy}{dt} + \phi\left(-\frac{1}{t}\right)\frac{1}{t^4}y = 0 \quad \text{........(21)},$$

or $$\frac{d^2y}{dt^2} + \phi\left(\frac{-1}{t}\right)\frac{1}{t^4}y = 0 \quad \text{............................(22)}.$$

And since, whether we take $x = -\dfrac{1}{t}$ or $x = \dfrac{1}{t}$, the middle term of (21) alike vanishes ($xt \pm 1 = 0$ satisfies that condition), we see, on replacing t by x in (22), that the solutions of (17) and of

$$\frac{d^2y}{dx^2} + \phi\left(\pm\frac{1}{x}\right)\frac{1}{x^4}\cdot y = 0 \quad \text{................(23)}$$

depend upon one another. Let $\phi(x) = ax^m$, then (17) becomes $f(m) = 0$, the solution of which is thus connected with that of $f(-m-4) = 0$. This verifies a result already given. Let $m = 2$, then the solution of $f(2) = 0$ is connected with that of $f(-2-4) = f(-6) = 0$. (Compare Questions 1854 and 1889.)

1990. (Proposed by Professor SYLVESTER.)—Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

Solution by Professor CAYLEY:

Some preliminary explanations are required in regard to this remarkable theorem.

1. I call to mind that a circular cubic (or cubic through the two circular points at infinity) has 16 foci, which lie 4 together on 4 different circles; and that the property of 4 concyclic foci is that taking any three of them A, B, C, the distances of a point P of the curve from these three foci are connected by a linear relation $\lambda \,.\, AP + \mu \,.\, BP + \nu \,.\, CR = 0$, where $\lambda \pm \mu \pm \nu = 0$, or if as is more convenient the distances are considered as $\pm$, then where $\lambda + \mu + \nu = 0$. A circular cubic may be determined so as to satisfy 7 conditions; having a focus at a given point is 2 conditions; hence a circular cubic may be determined so as to pass through three given points, and to have as foci two given points.

2. A Cartesian, or bicircular cuspidal quartic (that is a quartic having a cusp at each of the circular points at infinity) has nine foci, but of these there are three which lie in a line with the centre of the Cartesian (or intersection of the cuspidal tangents), and which are preeminently the foci of the Cartesian. We may, therefore, say that the Cartesian has three foci, which foci lie in a line, the axis of the Cartesian. A Cartesian may be determined to satisfy 6 conditions; having a focus at a given point is 2 conditions; but having for foci three given points on a line is 5 conditions; and hence a Cartesian may be found having for foci three given points on a line, and passing through a given point; there are in fact two such Cartesians, intersecting at right angles at the given point.

3. The theorem at first sight appears impossible; for take any three points F, G, H in a line and any other point A; then, as just remarked, there are,

having F, G, H for foci and passing through A, two Cartesians. And we may draw through F, G, H, and with A for focus, a circular cubic depending upon two arbitrary parameters; the position of a second focus of the circular cubic is (on account of the two arbitrary parameters) *primâ facie* indeterminate; and this is confirmed by the remark that the circular cubic can actually be so determined as to have for focus an arbitrary point B; and yet the theorem in effect asserts that the foci concyclic with A, of the circular cubic, lie on one or other of the two Cartesians.

4. To explain this, it is to be remarked that the arbitrary point B is a focus which is either concyclic with A or else not concyclic with A. In the latter case, although B is arbitrary, yet the foci concyclic with A may and in fact do lie on one of the Cartesians; the difficulty is in the former case if it arises; viz., if we can describe a cubic through the points F, G, H in a line, and with A and B as *concyclic* foci; that is, if we can find a third focus C, such that the distances from A, B, C of a point P on the curve are connected by a relation $\lambda:AP+\mu.BP+\nu.CP=0$, where $\lambda+\mu+\nu=0$. It may be shown that this is in a sense possible, but that the resulting cubic is not a proper circular cubic, but is the cubic made up of the line FGH taken twice, and of the line infinity. To show this, since the required cubic passes through the points F, G, H we have

$$\begin{aligned}\lambda.AF+\mu.BF+\nu.CF&=0\\ \lambda.AG+\mu.BG+\nu.CG&=0\\ \lambda.AH+\mu.BH+\nu.CH&=0\\ \lambda\quad+\mu\quad+\nu\quad&=0\end{aligned}\quad\text{and thence}\quad\begin{Vmatrix}AF, & AG, & AH, & 1\\ BF, & BG, & BH, & 1\\ CF, & CG, & CH, & 1\end{Vmatrix}=0,$$

being two conditions for the determination of the position of the point C; these give CG, CH as linear functions of CF; the distances CF, CG, CH of the point C from the points F, G, H in the line FGH are connected by a quadratic equation, and hence substituting for CG, CH their values in terms of CF, we have a quadratic equation for CF; as the given conditions are satisfied when C coincides with A or with B, the roots of this equation are CF=AF and CF=BF. But if CF=AF, then the linear relations give CG=AG and CH=AH, that is, C is a point opposite to A in regard to the line FGH. And similarly if CF=BF, then C is a point opposite to B in regard to the line FGH But C being opposite to A or B, the fourth concyclic focus D will be opposite to B or A; that is, the pairs A, B and C, D of concyclic foci lie symmetrically on opposite sides of the line FGH; this of course implies that the four points lie on a circle.

5. Taking $Y=0$ as the equation of the line FGH, $x^2+y^2-1=0$ as the equation of the circle through the four points A, B, C, D, then these lie on a proper cubic

$$(x^2+y^2+1)\,x+lx^2+ny^2=0$$

(not passing through the points F, G, H) and the four foci are given as the intersections with the circle $x^2+y^2-1=0$ of the pair of lines

$$x^2-2nx-nl=0.$$

But if we attempt to describe with the same four foci a cubic

$$(x^2+y^2+1)\,y+l'x^2+2m'xy+n'y^2=0,$$

then the foci are given as the intersections with the circle $x^2+y^2-1=0$ of the conic

$$y^2+2m'x-2l'y+m'^2-n'l'=0.$$

In order that these may coincide with the points (A, B, C, D) we must have

$$(x^2-2nx-nl)+(y^2+2m'x-2l'y+m'^2-n'l')=x^2+y^2-1;$$

that is, $\quad m'=n,\quad l'=0,\quad -nl+n^2-n'l'=-1.$

The last equation is $n'l' = n^2+1-nl$, which, assuming that nl is not equal to n^2+1, [in this case the cubic $(x^2+y^2+1)x+lx^2+my^2=0$ would reduce itself to the line and conic $(x+n)\left(x^2+y^2+\frac{x}{n}\right)=0$], since $l'=0$, gives $n'=\infty$, and therefore the cubic

$$(x^2+y^2+1)y+l'x^2+2m'xy+n'y^2=0,$$

reduces itself to $y^2=0$, that is, the cubic in question reduces itself to the line F, G, H, twice repeated, and the line infinity.

6. The conclusion is that F, G, H being given points on a line, and A and B being any other given points, there is not any proper cubic passing through F, G, H and having A, B for concyclic foci: and the *primâ facie* objection to the truth of the theorem is thus removed.

7. Considering the points F, G, H on a line and the point A as given, it has been seen that there are *two* Cartesians through A with the foci F, G, H; and the theorem asserts that in the circular cubics through F, G, H with the focus A, the foci concyclic with A lie on one or other of the two Cartesians: there are consequently through F, G, H with the focus A two systems of circular cubics corresponding to the two Cartesians respectively, each system depending upon two arbitrary parameters. But if we attend only to one of the two Cartesians and to the corresponding system of cubics, then the Cartesian passes through the four foci of each cubic, and if (instead of taking as given the points F, G, H and the focus A) we take as given the four concyclic foci A, B, C, D of a cubic, the theorem asserts that we have through A, B, C, D a Cartesian depending on two arbitrary parameters (or having for its axis an arbitrary line), and such that the foci of the Cartesian are the points of intersection F, G, H of its axis with the cubic. And I proceed to the proof of the theorem in this form.

8. The equation of a circular cubic having four foci on the circle $x^2+y^2-1=0$ is

$$(x^2+y^2+1)(Px+Qy)+lx^2+2mxy+ny^2=0;$$

and this being so, the four foci are the intersections of the circle with the conic

$$(Qx-Py)^2+2(-nP+mQ)x+2(mP-lQ)y+m^2-nl=0.$$

9. The general equation of a Cartesian is

$$(x^2+y^2+2Ax+2By+C)^2+2Dx+2Ey+F=0,$$

and by assuming for A, B, C, D, E, F the following values which contain the two arbitrary parameters α and θ, viz., by writing

$$2A=\theta Q,\ 2B=-\theta P,\ C=\alpha-1,\ D=-n\theta^2P+(m\theta^2-\alpha\theta)Q,$$
$$E=(m\theta^2+\alpha\theta)P-l\theta^2Q,\ F=-\alpha^2+\theta^2(m^2-nl),$$

we have the equation of a system (the selected one out of two systems) of Cartesians through the four foci; in fact, substituting the foregoing values, the equation of the Cartesian is

$$\left\{x^2+y^2+\theta(Qx-Py)+\alpha-1\right\}^2-2\alpha\theta(Qx-Py)$$
$$+2\theta^2(-nP+mQ)x+2\theta^2(mP-lQ)y-\alpha^2+\theta^2(m^2-nl)=0,$$

and writing herein $x^2+y^2-1=0$, the equation reduces itself to

$$\theta^2\left\{(Qx-Py)^2+2(-nP+mQ)x+2(mP-lQ)y+m^2-nl\right\}=0,$$

verifying that the Cartesian passes through the four foci.

The coordinates of the centre of the Cartesian are $x=-A$, $y=-B$, and the equation of its axis is $E(x+A)-D(y+B)=0$; we have therefore to show that the points of intersection of this line with the cubic are the foci of the Cartesian.

10. To find where the line in question meets the cubic

$$(x^2+y^2+1)(\mathrm{P}x+\mathrm{Q}y)+lx^2+2mxy+ny^2=0,$$

writing in this equation $x=-\mathrm{A}+\mathrm{D}\Omega,\quad y=-\mathrm{B}+\mathrm{E}\Omega,$
we have for the determination of Ω the equation

$$\left\{\mathrm{A}^2+\mathrm{B}^2+1-2(\mathrm{AD}+\mathrm{BE})\,\Omega+(\mathrm{D}^2+\mathrm{E}^2)\,\Omega^2\right\}\times$$
$$\left\{-\mathrm{AP}-\mathrm{BQ}+(\mathrm{DP}+\mathrm{EQ})\,\Omega\right\}+(l,\,m,\,n)\,(-\mathrm{A}+\mathrm{D}\Omega,\ -\mathrm{B}+\mathrm{E}\Omega)^2=0,$$

or observing that we have $\mathrm{AP}+\mathrm{BQ}=0$, this equation becomes

$$\begin{array}{lll} & (\mathrm{D}^2+\mathrm{E}^2)(\mathrm{DP}+\mathrm{EQ})\,\Omega^3 & \\ +\Big\{ & -2(\mathrm{AD}+\mathrm{BE})(\mathrm{DP}+\mathrm{EQ}) & +l\mathrm{D}^2+2m\mathrm{DE}+n\mathrm{E}^2\Big\}\,\Omega^2 \\ +\Big\{ & (\mathrm{A}^2+\mathrm{B}^2+1)(\mathrm{DP}+\mathrm{EQ}) & -2l\mathrm{AD}-2m(\mathrm{AE}+\mathrm{BD})-2n\mathrm{BE}\Big\}\,\Omega \\ +\Big\{ & & l\mathrm{A}^2+2m\mathrm{AB}+n\mathrm{B}^2\Big\}=0. \end{array}$$

11. Substituting for A, B, D, E their values in terms of $(\mathrm{P},\mathrm{Q},\alpha,\theta)$, we find

$$\begin{aligned} \mathrm{DP}+\mathrm{EQ} &= -\theta^2(n\mathrm{P}^2-2m\mathrm{PQ}+l\mathrm{Q}^2) \\ l\mathrm{A}^2+2m\mathrm{AB}+n\mathrm{B}^2 &= \tfrac{1}{4}\theta^2(n\mathrm{P}^2-2m\mathrm{PQ}+l\mathrm{Q}^2) \\ l\mathrm{AD}+m(\mathrm{AE}+\mathrm{BD})+n\mathrm{BE} &= -\tfrac{1}{2}\alpha\theta^2(n\mathrm{P}^2-2m\mathrm{PQ}+l\mathrm{Q}^2) \\ l\mathrm{D}^2+2m\mathrm{DE}+n\mathrm{E}^2 &= \left((nl-m^2)\theta^4+\alpha^2\theta^2\right)(n\mathrm{P}^2-2m\mathrm{PQ}+l\mathrm{Q}^2), \end{aligned}$$

and substituting these values in the equation for Ω, the whole equation divides by $\theta^2(n\mathrm{P}^2-2m\mathrm{PQ}+l\mathrm{Q}^2)$, and it then becomes

$$4(\mathrm{D}^2+\mathrm{E}^2)\,\Omega^3+4\left\{-2(\mathrm{AD}+\mathrm{BE})-(nl-m^2)\,\theta^2-\alpha^2\right\}\Omega^2$$
$$+4\left\{\mathrm{A}^2+\mathrm{B}^2+1-\alpha\right\}\Omega-1=0,$$

or, putting for shortness

$$\begin{aligned} \mathrm{C}' &= \mathrm{C}-\mathrm{A}^2-\mathrm{B}^2 &&= \alpha-1-\mathrm{A}^2-\mathrm{B}^2 \\ \mathrm{F}' &= \mathrm{F}-2(\mathrm{AD}+\mathrm{BE}) &&= -\alpha^2-\theta^2(nl-m^2)-2(\mathrm{AD}+\mathrm{BE}), \end{aligned}$$

the equation in Ω is

$$4(\mathrm{D}^2+\mathrm{E}^2)\,\Omega^3+4\mathrm{F}'\Omega^2-4\mathrm{C}'\Omega-1=0,$$

so that, Ω satisfying this equation, the intersections of the axis with the cubic are given by $x=-\mathrm{A}+\mathrm{D}\Omega,\ y=-\mathrm{B}+\mathrm{E}\Omega.$

12. The equation of the Cartesian, writing therein $x+\mathrm{A}=u$ and $y+\mathrm{B}=v$, and attending to the values of C' and F', is

$$(u^2+v^2+\mathrm{C}')^2+2\mathrm{D}u+2\mathrm{E}v+\mathrm{F}'=0.$$

And to find the foci, writing in this equation $u+\rho,\ v+i\rho$ in place of $u,\ v$, we find $\left\{u^2+v^2+\mathrm{C}'+2(u+vi)\,\rho\right\}^2+2(\mathrm{D}+\mathrm{E}i)\,\rho+2\mathrm{D}u+2\mathrm{E}v+\mathrm{F}'=0,$

that is, $(u^2+v^2+\mathrm{C}')^2+2\mathrm{D}u+2\mathrm{E}v+\mathrm{F}'$

$$+\left\{2(u+vi)(u^2+v^2+\mathrm{C}')+\mathrm{D}+\mathrm{E}i\right\}2\rho+4(u+vi)^2\rho^2=0.$$

Expressing that this equation in ρ has two equal roots, we find

$$4(u+vi)^2\left\{(u^2+v^2+\mathrm{C}')^2+2\mathrm{D}u+2\mathrm{E}v+\mathrm{F}'\right\}$$
$$-\left\{2(u+vi)(u^2+v^2+\mathrm{C}')+\mathrm{D}+\mathrm{E}i\right\}^2=0,$$

that is, $4(2\mathrm{D}u+2\mathrm{E}v+\mathrm{F}')(u+vi)^2$

$$-4(u^2+v^2+\mathrm{C}')(u+vi)(\mathrm{D}+\mathrm{E}i)-(\mathrm{D}+\mathrm{E}i)^2=0,$$

which equation is in fact the equation of the three tangents from one of the circular points at infinity. Writing it under the form $Y + Vi = 0$, the nine foci of the Cartesian are given as the intersections of the two cubics $U = 0$, $V = 0$. But of these nine points, three, the foci that we are concerned with, lie on the axis, or line $Eu - Dv = 0$; in fact, we have

$$U = 4(u^2 - v^2)(2Du + 2Ev + F') - 4(uD - vE)(u^2 + v^2 + C') - (D^2 - E^2),$$

$$V = 8uv(2Du + 2Ev + F') - 4(uE + vD)(u^2 + v^2 + C') - 2DE;$$

and hence $2DEU - (D^2 - E^2)V$

$$= (Eu - Dv)\left\{8(Du + Ev)(2Du + 2Ev + F') - 4(D^2 + E^2)(u^2 + v^2 + C')\right\} = 0,$$

which shows that the nine points lie three of them on the line $Eu - Dv = 0$, and the remaining six on the conic

$$2(Du + Ev)(2Du + 2Ev + F') - (D^2 + E^2)(u^2 + v^2 + C') = 0.$$

13. We have thus the three foci given as the intersections of the axis $Eu - Dv = 0$, with the cubic

$$U = 4(u^2 - v^2)(2Du + 2Ev + F') - 4(uD - vE)(u^2 + v^2 + C') - (D^2 - E^2) = 0;$$

or, writing in this last equation $u = D\Omega$, $v = E\Omega$, that is $x = -A + D\Omega$, $y = -B + E\Omega$, we have

$$u^2 - v^2 = (D^2 - E^2)\Omega^2, \quad uD - vE = (D^2 - E^2)\Omega.$$

The whole equation divides by $(D^2 - E^2)$, and omitting this factor, it is

$$4\Omega^2\left\{2(D^2 + E^2)\Omega + F'\right\} - 4\Omega\left\{(D^2 + E^2)\Omega^2 + C'\right\} - 1 = 0,$$

that is
$$4(D^2 + E^2)\Omega^3 + 4F'\Omega^2 - 4C'\Omega - 1 = 0,$$

the same equation as the equation in Ω before obtained; that is the intersections of the cubic with the axis are the three foci of the Cartesian.

1974. (Proposed by C. M. Ingleby, LL.D.)—If $x^2 = y^2 + z^2$, show that it can furnish no numerical formula which is not contained in the identical equation $\left(a + \frac{1}{a}\right)^2 \equiv \left(a - \frac{1}{a}\right)^2 + 4$.

I. *Solution by* Samuel Bills.

Take the given formula and put it in the form $(x + y)(x - y) = z^2$. Now it is very obvious that the quantity $x + y$ can have no value but may be expressed in the form $x + y = az$, and then we should have $x - y = \frac{z}{a}$. From these two equations we find $x = \frac{1}{2}\left(a + \frac{1}{a}\right)z$, $y = \frac{1}{2}\left(a - \frac{1}{a}\right)z$. Substituting these results in the given formula, and dividing by z^2, we obtain $\left(a + \frac{1}{a}\right)^2 = \left(a - \frac{1}{a}\right)^2 + 4$; hence the truth of the proposition.

II. *Solution by the* PROPOSER.

If $x^2 = y^2 + z^2$, multiply each term by 4 and divide by z^2; then $\left(\frac{2x}{z}\right)^2 = \left(\frac{2y}{z}\right)^2 + 4$; and putting $(x+y)+(x-y)$ and $(x+y)-(x-y)$ for $2x$ and $2y$ respectively, we get

$$\left(\frac{x+y}{z} + \frac{x-y}{z}\right)^2 = \left(\frac{x+y}{z} - \frac{x-y}{z}\right)^2 + 4,$$

and this becomes identical, if $\frac{x+y}{z}$ and $\frac{x-y}{z}$ are regarded as reciprocals of each other; *i.e.* if $\frac{x+y}{z} \cdot \frac{x-y}{z} = 1$; which is true if $x^2 - y^2 = z^2$. Therefore, if we put $\frac{x+y}{z} = a$ and $\frac{x-y}{z} = \frac{1}{a}$, we get $\left(a + \frac{1}{a}\right)^2 = \left(a - \frac{1}{a}\right)^2 + 4$; and for every triad of values for x, y, z, satisfying the original equation, there is a corresponding value of the ratio a, making the latter equation coincide with the former.

COR.—Every case is given by four sets of values for x, y, z, or a.

EX.—If $a = 2, \frac{1}{2}, 3$, or $\frac{1}{3}$, we get $5^2 = 4^2 + 3^2$. If $a = 4, \frac{1}{4}, \frac{5}{3}$ or $\frac{3}{5}$, we get $17^2 = 15^2 + 8^2$; and so forth. These values are determined by the conjugate equations, $a = \frac{x+y}{z}$, or $a = \frac{x+z}{y}$; and $\frac{1}{a}$ gives the same results as a, because $x^2 - (y^2 + z^2)$ is a reciprocal function of a.

1983. (Proposed by W. LEA.)—Find the roots of the equation $x^3 + px^2 + qx + r = 0$, the ratio of any two of these roots being given.

Solution by the REV. ROBERT HARLEY, F.R.S.

Let ρ be the given ratio; then, since

$$x^3 + px^2 + qx + r = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

we have

$$\rho^3 x^3 + p\rho^2 x^2 + q\rho x + r = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now $(1)-(2)$ and $\rho^3(1)-(2)$ give, after slight reduction,

$$(1+\rho+\rho^2)\,x^2 + p\,(1+\rho)\,x + q = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$p\rho^2 x^2 + q\rho\,(1+\rho)\,x + r\,(1+\rho+\rho^2) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

And $p\rho^2(3) - (1+\rho+\rho^2)(4)$ gives

$$\rho\,(1+\rho)\left\{p^2\rho - q\,(1+\rho+\rho^2)\right\}x + pq\rho^2 - r\,(1+\rho+\rho^2)^2 = 0;$$

or

$$x = \frac{1}{\rho\,(1+\rho)} \cdot \frac{r\,(1+\rho+\rho^2)^2 - pq\rho^2}{p^2\rho - q\,(1+\rho+\rho^2)}, \text{ one root,}$$

therefore $\rho x = \dfrac{1}{1+\rho} \cdot \dfrac{r(1+\rho+\rho^2)-pq\rho^2}{p^2\rho-q(1+\rho+\rho^2)}$, a second root, and

$-p-(1+\rho)x = \dfrac{1}{\rho} \cdot \dfrac{-p^3\rho^2+pq\rho(1+\rho)^2-r(1+\rho+\rho^2)^2}{p^2\rho-q(1+\rho+\rho^2)}$, the third root.

NOTE.—These expressions may of course be exhibited in a variety of forms, because a relation exists between the ratio ρ and the coefficients p, q, r.

That relation, obtained by the elimination of x is as follows, viz ,

$$\begin{aligned} r^2\rho^6-(pqr-3r^2)\rho^5+(p^3r-5pqr+q^3+6r^2)\rho^4 \\ +(2p^3r-p^2q^2-6pqr+2q^3+7r^2)\rho^3 \\ +(p^3r-5pqr+q^3+6r^2)\rho^2-(pqr-3r^2)\rho+r^2=0, \end{aligned}$$

a recurring equation, which is, as might be expected, of the sixth degree in ρ. The six values of ρ, considered as a function of the roots x_1, x_2, x_3, are $\dfrac{x_1}{x_2}$, $\dfrac{x_1}{x_3}$, $\dfrac{x_2}{x_3}$, and their reciprocals.

The above equation may be put under the form

$$r^2\left(\rho+\frac{1}{\rho}\right)^3-(pqr-3r^2)\left(\rho+\frac{1}{\rho}\right)^2+(p^3r-5pqr+q^3+3r^2)\left(\rho+\frac{1}{\rho}\right)$$
$$+2p^3r-p^2q^2-4pqr+2q^3+r^2=0 \quad\ldots\ldots\ldots\ldots\ldots (5).$$

The condition that (1) may have two equal roots, obtained from (5) by making $\rho+\dfrac{1}{\rho}=2$, is $4p^3r-p^2q^2-18pqr+4q^3+27r^2=0$,

the first member of which is (to a factor *près*) the discriminant of (1).

This is known to be true from other considerations.

II. *Solution by* R. BALL, M.A.

If k be the ratio of two roots α and β of the cubic $(a, b, c, d)(x, 1)^3=0$, we have the equations

$$\alpha-k\beta=0,\quad \alpha+\beta+\gamma+\frac{3b}{a}=0,\quad \alpha\beta+\alpha\gamma+\beta\gamma-\frac{3}{a}=0.$$

From the solution of these equations we obtain

$$\left.\begin{aligned} \alpha &= \frac{-6ck}{3b(1+k)\mp\sqrt{\{9b^2(1+k)^2-12ac(1+k+k^2)\}}} \\ \beta &= \frac{-6c}{3b(1+k)\mp\sqrt{\{9b^2(1+k)^2-12ac(1+k+k^2)\}}} \\ \gamma &= -3\frac{b}{a}+\frac{6c+6ck}{3b(1+k)\mp\sqrt{\{9b^2(1+k)^2-12ac(1+k+k^2)\}}} \end{aligned}\right\}\ldots(\text{I}).$$

It remains to account for the ambiguity of sign. This arises from the circumstance that α, β, γ have not been determined to satisfy the cubic, but merely the three equations from which they were derived. To be the roots of the cubic they must further make $\alpha\beta\gamma=-\dfrac{d}{a}$, and that sign must be attached to the radical which will fulfil this condition. One, then, of the

signs must, if k be really the ratio of two roots, satisfy this; but, it may be inquired of what cubic are the three values of α, β, γ the roots, if the other sign be given to the radical? The answer is not difficult.

Let $(\alpha-k\beta)(\beta-k\alpha)(\alpha-k\gamma)(\gamma-k\alpha)(\beta-k\gamma)(\gamma-k\beta)$ be expanded and their values substituted for the symmetric functions: the result must equal zero, since we have assumed that k is the ratio of a pair of roots. If in the equation thus produced k be regarded as variable, the roots are the ratios of the roots of the given cubic. This equation may therefore be found by the elimination of α between

$$\begin{cases} \alpha = \dfrac{-6ck}{3b(1+k) \mp \sqrt{\{9b^2(1+k)^2-12ac(1+k+k^2)\}}} \\ a\alpha^3+3b\alpha^2+3c\alpha+d=0, \end{cases}$$

or since it is reciprocal it may be obtained more easily by forming the symmetric functions which are its coefficients. The result is

$$\begin{aligned} &a^2d^2k^6 + (3a^2d^2-9abcd)k^5 + (6a^2d^2-45abcd+27ac^3+27b^3d)k^4 \\ &+(7a^2d^2-54abcd+54ac^3-81b^2c^2+54b^3d)k^3 \\ &+(6a^2d^2-45abcd+27ac^3+27b^3d)k^2+(3a^2d^2-9abcd)k+a^2d^2=0 \ \ldots\ldots (\text{II}). \end{aligned}$$

This is easily verified by the consideration that if the cubic have equal roots k must equal unity; but if in this k be made $=1$, the result becomes

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd=0,$$

the well known condition for equal roots.

Since, however, k is a given quantity a, b, c, d must be such as to satisfy (II); and since d enters in the second degree, there are two values of d which (a, b, and c remaining the same) will accomplish this. One of these values for d must, of course, be that given in the cubic; call the other d'. Then we find that there are two cubics, namely,

$$(a, b, c, d)(x, 1)^3=0, \ \text{ and } \ (a, b, c, d')(x, 1)^3=0,$$

whose roots satisfy the system

$$\alpha-k\beta=0, \ \alpha+\beta+\gamma+\frac{3b}{a}=0, \ \beta\gamma+\gamma\alpha+\alpha\beta-\frac{3c}{a}=0.$$

The upper sign then in the values (I) constitutes them the roots of one of these equations, the lower sign gives the roots of the other.

Suppose that k is not the ratio of two of the roots of $(a, b, c, d)(x, 1)^3=0$, of what cubics are the systems (I) the roots? Here k and d do not then fulfil the relation (II), but by solving (II) as a quadratic for d, two values d' and d'' are found, and the cubics required are

$$(a, b, c, d')(x, 1)^3=0, \qquad (a, b, c, d'')(x, 1)^3=0.$$

This remark may be stated generally. Suppose a rational and integral relation is given among the roots of a cubic $f(\alpha, \beta, \gamma)=0$ in which the roots enter to the nth degree. If we can solve

$$f(\alpha, \beta, \gamma)=0, \quad \alpha+\beta+\gamma+\frac{3b}{a}=0, \quad \beta\gamma+\gamma\alpha+\alpha\beta-\frac{3c}{a}=0,$$

$2n$ systems are found; one of these systems are roots of the given cubic: to what cubics do the others belong?

$f(\alpha, \beta, \gamma).f(\alpha, \gamma, \beta).f(\beta, \alpha, \gamma).f(\beta, \gamma, \alpha).f(\gamma, \alpha, \beta).f(\gamma, \beta, \alpha)=0$:— evaluate this symmetric function in terms of the coefficients, the "weight"

of the result is $6n$; but the weight of d is 3, therefore d will enter in the power of $2n$, consequently $2n$ values of d will satisfy this result; if these values be $d', d'', d''', d'''^{.2n}$, the system of cubics is

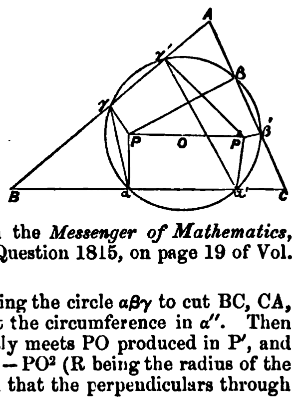
$$(a, b, c, d')\,(x, 1)^3 = 0,\ \ (a, b, c, d'')\,(x, 1)^3 = 0,\ (a, b, c, d'''^{.2n})\,(x, 1)^3 = 0;$$

one of which will, of course, be identical with the given cubic.

1975. (Proposed by F. D. THOMSON, M.A.)—Prove the following construction for finding the point whose trilinear coordinates are the reciprocals of those of a given point P. From P draw $P\alpha$, $P\beta$, $P\gamma$ perpendicular to the sides of the triangle of reference ABC; and let O be the centre of the circle round $\alpha\beta\gamma$: join PO and produce it to P′, making OP′ = PO: then P′ is the point required.

Solution by the PROPOSER.

Draw $P'\alpha'$, $P'\beta$, $P'\gamma'$ perpendicular to the sides. Then it is easily seen that the circle $\alpha\beta\gamma$ passes through $\alpha'\beta'\gamma'$. Join $\alpha\gamma$, $\alpha'\gamma'$. Then $B\alpha \,.\, B\alpha' = B\gamma \,.\, B\gamma'$, therefore $B\alpha : B\gamma = B\gamma' : B\alpha'$. Hence the triangles $B\gamma\alpha$, $B\alpha'\gamma'$ are similar, therefore $\angle B\gamma\alpha = B\alpha'\gamma'$, and $\angle \alpha\gamma P = \gamma'\alpha'P'$: hence the triangle $\gamma\alpha P$ is similar to $\alpha'\gamma'P'$, therefore $\gamma P : \alpha P = \alpha'P' : \gamma'P'$; and similarly $\alpha P : \beta P = \beta'P' : \alpha'P'$; therefore $\alpha P \,.\, \alpha'P' = \beta P \,.\, \beta'P' = \gamma P \,.\, \gamma'P'$, which proves the theorem.



[Mr. THOMSON remarks that the above was suggested by Mr. BESANT's paper in the *Messenger of Mathematics*, Vol. III., p. 222. See also the Solution of Question 1815, on page 19 of Vol. V. of the *Reprint*.

Mr. DALE proves the theorem by supposing the circle $\alpha\beta\gamma$ to cut BC, CA, AB in α', β', γ', and producing αP to meet the circumference in α''. Then the perpendicular to BC through α' obviously meets PO produced in P′, and $\alpha''P = \alpha'P'$; also $\alpha P \,.\, P\alpha'' = \alpha P \,.\, \alpha'P' = R^2 - PO^2$ (R being the radius of the circle $\alpha\beta\gamma$): and similarly it may be shown that the perpendiculars through β', γ' likewise pass through P′, and that

$$\alpha P \,.\, \alpha'P' = \beta P \,.\, \beta'P' = \gamma P \,.\, \gamma'P' = R^2 - PO^2.]$$

1896. (Proposed by Dr. WILSON.)—Show that the lines trisecting an angle of a triangle do not trisect the opposite side.

Solution by J. R. Allen; S. W. Bromfield; R. Tucker, M.A.; J. Dale; A. Renshaw; W. H. Laverty; J. H. Taylor, B.A.; *and many others.*

If the bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles (by Euc. vi. 3) and the bisector perpendicular to the side. If therefore the two trisectors of an angle of a triangle trisected the opposite side, these lines would be *both* perpendicular to that side.

[Mr. McCormick remarks that the proof may be effected by the aid of the *first* book of Euclid (as well as by Euc. vi., 3), and refers to Lardner's *Euclid*, Book i., Prop. 26, Cor. 1.]

1881. (Proposed by J. R. Allen.)—Let O be the middle point of any chord AA′ of a circle ABA′; through O draw any other chord BOB′; join AB, A′B′ and produce these lines to meet in P. Show that the locus of P is a straight line parallel to AA′; and is the same as that of the intersection of tangents drawn to the circle at B, B′.

Solution by J. H. Taylor, B.A.; A. Renshaw; *and others.*

Let R and R′ (Fig 1) be the intersections of pairs of tangents at A, A′ and B, B′. Join RR′.

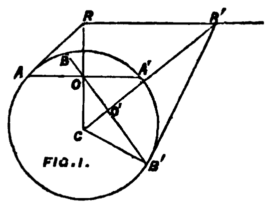


FIG. 1.

Then $CO \cdot CR = CA^2 = CB'^2 = CO' \cdot CR'$, therefore the triangles CO′O, CRR′ are similar, for they have a common angle and the sides about it proportional;
therefore $\angle CRR' = CO'O =$ a right angle,
therefore RR′ is the polar of O.

Next (Fig 2) let AB, A′B′ intersect in P, and AB′, A′B in P′; then, by a property of a complete quadrilateral, (P′. B′OA′P) is an harmonic pencil, and AO = OA′; therefore the transversal AA′ is parallel to P′P. Let B′B meet P′P in M; then (B′OBM) is an harmonic range; therefore M is on the polar of O: but P′P is parallel to AA′, therefore P′P is the polar of O: and it is also the locus of the intersection of tangents at B and B′, by the first part.

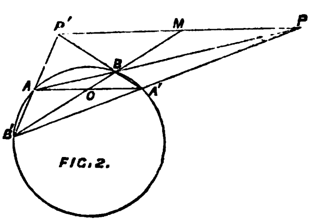
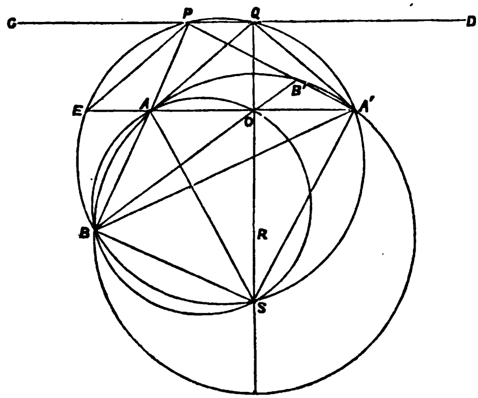


FIG. 2.

II. *Solution by the* PROPOSER; H. MURPHY; E. MCCORMICK; *and others.*



Through O draw the diameter ROQ; through P draw CD parallel to AA′, cutting the diameter ROQ in Q; round the points B, A, O draw a circle cutting the diameter ROQ in S; and join SA, SA′, SB.

Now, $\because \angle AOS$ = a right $\angle$ (Euc. iii. 3),

$\therefore \angle ABS$ = a right $\angle$ (Euc. iii. 22) $= OBA + OAS = OA'B' + OA'S$;

$\therefore$ the whole $\angle$ $PA'S = ABS$ = a right $\angle$.

Also, $\because$ CD is parallel to AA′ and SQ cuts AA′ at right $\angle$s,

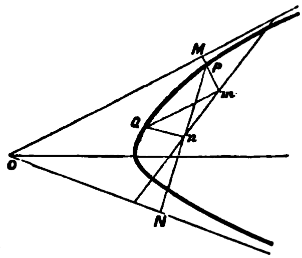
$\therefore \angle PQS$ = a right $\angle$ (Euc. i. 29),

therefore a circle may be drawn round the points P, Q, A′, S, B. Produce A′A to meet this circle in E, and join PE, QA, QA′, BA′.

Now, since PQ is parallel to EA′, it may easily be proved that PE = QA′ = QA, therefore QA is parallel to PE, therefore $\angle QAA' = PEA' = PBA'$, therefore QA is a tangent to the circle BAA′; hence the point Q and the line CD are given in position. Also it is well known that the locus of Q is the same as that of the intersections of tangents at B, B′; it may also be proved geometrically by drawing circles round QARA′ and BRB′.

1899. (Proposed by G. O. HANLON.)—If perpendiculars from any point P on a hyperbola are drawn to the asymptotes, prove that the line joining the feet of perpendiculars from any other point Q on the curve to these lines passes through a fixed point F. Also find the locus of F, as P varies on the curve, and determine the hyperbola for which this locus reduces to a point.

I. *Solution by* J. H. TAYLOR, B.A.; *the* PROPOSER; A. RENSHAW; *and others.*



Taking the asymptotes as axes, and putting ω for the angle of ordination MON, let the equation of the hyperbola be $xy = k^2$; and let (x', y'), and (x'', y'') be the coordinates of the points P and Q. Then the respective equations of the lines PN, Qn, PM, Qm, are

$$y-y' = (x'-x)\sec\omega,\ y = y'',$$
$$x-x' = (y'-y)\sec\omega,\quad x = x'',$$

Hence the equation of mn is

$$k^2\left\{yy' + xy'\cos\omega - \cos\omega\,(k^2+y'^2\cos\omega)\right\}$$
$$-y''\,(yy'^2\cos\omega + xy'^2 - y'^3\cos\omega - k^2y'\cos^2\omega) = 0,$$

which contains the indeterminate y'' and therefore passes through the point given by

$$yy' + xy'\cos\omega - \cos\omega\,(k^2+y'^2\cos\omega) = 0,$$
$$yy'\cos\omega + xy' - \cos\omega\,(y'^2+k^2\cos\omega) = 0;$$

that is, through the fixed point (F) whose coordinates are $(y'\cos\omega,\ x'\cos\omega)$. The locus of F when (x', y') varies is at once seen to be the hyperbola $xy = k^2\cos^2\omega$, which reduces to the origin if $\cos\omega = 0$, that is if the hyperbola is rectangular.

II. *Solution by* W. H. LAVERTY.

Let us represent by [X] the anharmonic ratio of any four points of a series of which X is one point, always supposing that if [X] = [Y], [Y] is the anharmonic ratio of four *corresponding* points of the second series.

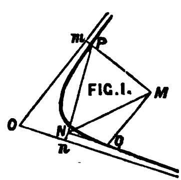


FIG. I.

Let P, Q (Fig. 1) be the two points on the conic; Pm, Pn; QM, QN; the perpendiculars from P and Q, on the asymptotes, and on Pm, Pn, respectively. Then $\angle$ NPM = mCn; and if P be fixed, we have [M] = [N]; moreover, these homographic series are in perspective, therefore MN envelopes a point, *i.e.* always passes through a fixed point.

Next, having shown that MN always passes through a fixed point, we may, to solve the latter part of the problem, take any two points Q, Q, keeping which points fixed, we find the locus of the intersection of MN and M′N′ as P varies. Take then the points at infinity on the asymptotes. Then MN, M′N′, become respectively the lines through m, n, parallel to Pn, Pm, (Fig. 2); let their intersection be R.

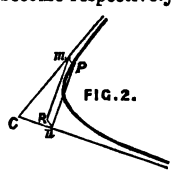
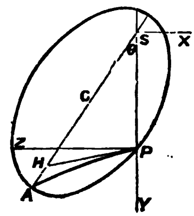


FIG. 2.

Now, P[m] = P[n], therefore [m] = [n]; and these homographic series are not in perspective, therefore mn envelopes a conic. Again, in the triangle Rmn; m and n always lie on fixed lines; Rm, Rn, (being perpendicular to the asymptotes) always pass through fixed points at infinity, and mn envelopes a conic, therefore the locus of R is a conic, which evidently degenerates into a point (the origin) when the hyperbola is equilateral.

1722. (Proposed by R. TUCKER, M.A.)—A perfectly elastic ball is dropped from the fixed focus of a perfectly smooth ellipse; supposing the position of the ellipse to vary, find the locus of the vertices of the curves described after impact on the elliptic arc.

Solution by the PROPOSER.



Take one position of the ellipse, of which S, H are the foci and C, A the centre and vertex respectively. Let the chord of descent SP ($=r$) make the $\angle$ ASP $=\theta$ with the major axis, and through P draw the horizontal line PZ: then since the ball is perfectly elastic, it may, after impact, be considered as projected in the direction PH with the velocity acquired in SP; hence, putting $\epsilon = \angle$ HPZ, the equation to its path, referred to horizontal and vertical axes SX, SY through S is

$$r-y = x\tan\epsilon - \frac{x^2}{4r\cos^2\epsilon}.$$

The coordinates of the vertex are

$$X = r\sin 2\epsilon,\ \ Y = r\cos^2\epsilon,$$

whence, combining with the equations of condition

$$2a-r = 2ae\sin\theta\sec\epsilon,\ \ c = r(1-e\cos\theta),$$

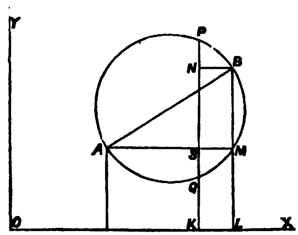
we obtain for the locus required

$$(X^2+4Y^2)^3-16a(X^2+4Y^2)^2\left\{Y+ae^2+a\right\}+64a^2Y(X^2+4Y^2)(Y+2c)$$
$$+256a^2c^2Y^2 = 0.$$

1963. (Proposed by M. W. CROFTON, B.A.)—Show that the equation of a circle on the line joining (x', y') and (x'', y'') as diameter is

$$(x-x')(x-x'')+(y-y')(y-y'')=0.$$

Solutions by T. J. SANDERSON, B.A.; H. TOMLINSON; REV. R. H. WRIGHT, M.A.; J. DALE; R. TUCKER, M.A.; S. W. BROMFIELD; A. COHEN, B.A.; REV. J. L. KITCHIN, M.A.; W. H. LAVERTY; *the* PROPOSER; *and many others.*



1. Let A, B, be the points (x', y'), (x'', y'') respectively; P, or (x, y), any point on the circle described on AB as diameter. Draw the ordinate PK meeting the circle again in Q, and the ordinate BL meeting the circle again in M. Join AM, which will be perpendicular to BL, since AB is a diameter. Let AM meet PK in S. Then, if BN be perpendicular to PQ, we have AS . SM = PS . SQ = PS . PN (if BN be perpendicular to PQ),

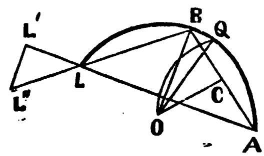
or $$(x-x')(x''-x) = (y-y')(y-y''),$$

that is $$(x-x')(x-x'')+(y-y')(y-y'') = 0,$$ the equation required.

2. *Otherwise:* The given equation manifestly represents a circle, since the coefficients of the highest powers are the same, and this circle passes through the points (x', y'), (x'', y''), (x', y''), (x'', y'); and these are the angular points of a rectangle, hence the equation represents a circle on the diagonal formed by joining (x', y') with (x'', y'') as diameter.

1759. (Proposed by A. Renshaw.)—Travelling at night on a line of railway of circular form, I noticed three distant lights L, L′, L″. At a certain station A, the lights L and L′ appeared to coalesce; and at the next station B, the lights L and L″ assumed the same appearance; the distances LL′, LL″, L′L″ are k, l, m, respectively, the direct distance from A to B is n, and the circle of which the Railway forms a part would if completed pass through L. Now if with radius of this circle and centre A, another circle were described, cutting the Railway between A and B in Q, find the distance along the line from Q to B.

Solution by E. L. Rosolis; *the* Proposer; *and others.*



From O, the centre of the circle ABL, draw OC perpendicular to AB; then, putting $\angle L'LL'' = ALB = BOC = \alpha°$, this angle is given by the relation $2kl \cos\alpha = k^2 + l^2 - m^2$; moreover $OB = \frac{1}{2}n \operatorname{cosec}\alpha$; and since the arc AQ subtends an angle of 60° at O, we have $\angle BOQ = 2\alpha° - 60°$; and consequently arc $BQ = \dfrac{\alpha° - 30°}{180}\pi n \operatorname{cosec}\alpha$.

1900. (Proposed by A. Renshaw.)—The Austrian Government have lately issued a loan of 734,694 Bonds of £19 17*s.* 0*d.* sterling, or 500 francs, or 200 florins Austrian, value in silver; and it appears that a contract for it has been entered into between the Imperial Government of Austria and the Comptoir d'Escompte of Paris, in combination with several capitalists. The Bonds will be issued at £13. 14*s.* 6*d.* each, with coupons attached, payable half-yearly, of the value of 9*s.* 11*d.* each, being at the rate of 5 per cent. per annum on the par value of £19. 17*s.* 0*d.* from the 1st December, 1865. They will be redeemed in 37 years by half-yearly drawings, to take place publicly, at the Austrian Embassy in Paris, on the 1st May and 1st November of each year. At each drawing an equal number of Bonds, viz. 9,928, will be withdrawn and paid of at par (£19. 17*s.* 0*d.*) with the half-yearly dividend. Find, from these data, the rate of interest at which the Austrian Government are thus borrowing.

Solution by SAMUEL BILLS.

In the first place, the sum actually received by the Austrian Government on account of the loan will be

$$£13.\ 14s.\ 6d. \times 734694 = £10,083,675.\ 3s.\ 0d. \ldots\ldots\ldots (A).$$

The amount paid back to the Bondholders over and above the principal, will be composed of two parts. The first part will be the difference between £19. 17s. 0d. and £13. 14s 6d. multiplied by 734694, that is,

$$£6.\ 2s.\ 6d. \times 734694 = £4,499,820.\ 15s.\ 0d. \ \ldots\ldots\ldots (B).$$

The second part will be composed of the amount of the coupons paid at the several half-yearly drawings. Put $734694 = a$, and $9928 = b$; then the several amounts paid at the 1st, 2nd, 3rd,.... 73rd, 74th drawings will be

$$a\,(9s.\ 11d.),\ (a-b)\,(9s.\ 11d.),\ (a-2b)\,(9s.\ 11d.)\ldots.(a-72b)\,(9s.\ 11d.),$$
$$9950\,(9s.\ 11d.).$$

The truth of the last will appear by considering that

$$73 \times 9928 + 9950 = 734694.$$

Now, the sum of the first 73 terms of the preceding series is

$$73\,(a-36b)\,(9s.\ 11d.) = £13,656,181.\ 3s.\ 6d.$$

Also, 9950 (9s. 11*d.*) = £4933. 10s. 10d. Adding these together, we have, as the total paid on account of the coupons,

$$£13,661,114.\ 14s.\ 4d. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C).$$

Adding together (B) and (C), we obtain, as the total amount paid by the Austrian Government to the Bondholders, over and above the amount of the principal,

$$£18,160,935.\ 9s.\ 4d. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (D).$$

Let x denote the rate per £. per half year; then, putting £13. 14s. 6d. $=p$, we shall have the respective amounts of interest corresponding to the 1st, 2nd, 3d,......73rd, last half-years,

$$apx,\quad (a-b)\,px,\quad (a-2b)\,px,\ \ldots\ldots(a-72b)\,px,\quad 9950px.$$

Hence, by addition, we should have

$$73\,(a-36b)\,px + 9950px = (D).$$

Restoring the values of a, b, p and (D), we should find £9. 12s. $1\frac{1}{4}$d. nearly as the rate per cent. per annum required.

OBSERVATIONS ON THE "THREE AND FOUR POINT PROBLEMS" IN THEIR RELATIONS TO INFINITY. BY W. S. B. WOOLHOUSE, F.R.A.S.

The three point problem, first proposed by me in the *Lady's and Gentleman's Diary* for 1861, and reproduced in the *Educational Times* for December, 1862, as Question 1333, was ennunciated thus:—

"Three points being taken at random in space as the corners of a plane triangle, determine the probability that it shall be acute."

In the Solutions that have been given to this problem, the three points are first conceived to be limited to the volume of a given sphere; and as the

resulting value of the probability, viz., $\frac{33}{70}$, is obviously independent of the magnitude of the sphere, it is assumed to apply to the extreme case in which the radius is supposed to be infinite.* According to this process, unlimited space is represented by a sphere of infinite radius, to which conformation the mind would seem to be naturally led by an idea of symmetry in all directions round a finite region. Other reasons of a practical nature might be adduced in support of the general consistency of this hypothesis, with reference to the subject under consideration, giving to it, in some respects at least, a preference to what might otherwise be regarded as more strictly legitimate in the abstract. My mind is fully made up on this point, and I may be induced to discuss it hereafter. There is one thing quite certain, that whatever hypothesis is taken ought to be consistent with itself. Also, to comprehend every combination of the associated points, it is an essential condition that they shall severally and alike occupy every assignable position.

The present communication is for the purpose of briefly adverting to the untenable character of certain statements relative to this problem, contained in an interesting paper by Professor De Morgan, "On Infinity, &c.," printed in the *Transactions of the Cambridge Philosophical Society*, Vol. XI., Part I., page 3. It is probable that Professor De Morgan may have become aware of the logical deficiency of the reasoning advanced by him, if he has since given the subject any attention. Under any circumstances, I am assured that an earnest endeavour to throw some additional light on a subject, the difficulties of which are so critical and so peculiarly calculated to create differences of opinion, will be received with his characteristic liberality and independence of thought. To avoid, if possible, the risk of being obscure, I shall first transcribe, *in extenso*, the statements in question, which are the following:—

"The absolute infinite is avoided by recourse to increase without limit. If, for instance, we have to choose points in space which shall satisfy certain conditions, and if we first choose them within a given sphere, and then increase the radius of the sphere without limit, do we not finally allow unlimited choice? What point of space is omitted out of a sphere of infinite radius? Certainly not any assignable point. But, on the other hand, we know well that we dare not deny of an infinite sphere anything which is true of *any sphere however great:* if there be anything which is true, and is equally true of all the increasing spheres, that truth is, with obvious reason and invariable success, predicated of the sphere increased *ad infinitum*. Now it is certainly true of any sphere however great, that there is infinitely more space outside than inside. If this be also true of an infinite sphere, that sphere does not include all space: if it be false, where does the sphere begin to include what it does not include? Where does the ever remaining external space lose that character? Let us see which of the two assertions will a problem justify."

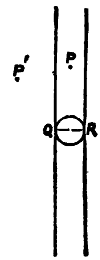
"Three points are taken at hazard in space, all points being equally likely: what is the chance of the triangle having three acute angles? I saw this problem solved in a mathematical journal, by a sound use of the integral calculus, on the supposition of the points being taken within a given sphere. The answer gave a finite chance for an acute-angled triangle, which remained finite when the sphere was enlarged *ad infinitum*. But it is very easily shown that the chance of an acute-angled triangle must be infinitely small. Take any one base; at its ends, draw two planes perpendicular to it, and infinitely extended: also, on the base as a diameter, describe a sphere. An

* The *Reprint*, Vol. IV., page 80, contains a note from me, respecting an elementary property, that the mathematical value of the probability is unaltered when one of the points is restricted to a fixed position on the surface of the sphere.

acute-angled triangle must have its vertex within the infinite strip between the parallel planes, and outside the sphere. Now it is clear that the possible vertices outside the strip infinitely outnumber those within the strip, if points be equally distributed through space. For any given base, and consequently for any number of bases, there is no appreciable chance of an acute-angled triangle: the same then for all bases and unlimited choice, if any one base be as likely as any other."

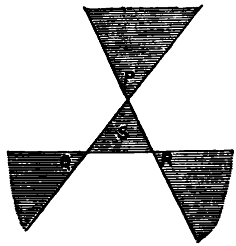
The following objections arise upon these statements. In the first place, it may be asserted that a sphere of infinite radius can have no existing boundary, if we accept the term infinite in an absolute sense; which we must do here, as the sphere is specifically adopted as the representative of infinite space, and does not admit any consideration of different orders of infinity. An infinitely distant boundary is equivalent to no boundary at all, since any conception whatever of boundary would instantly negative the idea of an infinite sphere. Again, when a sphere is taken and accepted as the representative of infinite space, it is evident that an inconsistency amounting to no less than a direct contradiction, is immediately introduced by any assumption of the existence of points outside it.

In the second paragraph, it is professed to be shown, that the chance of an acute triangle is infinitely small instead of a determinate finite value. The fallacy, however, is introduced by taking an assumed base QR, and always treating it as finite, and at the same time allowing the vertex P to roam uncontrolled over infinite space, both within and without the strip in the manner described. Such constructions are necessarily partial, defective, and therefore erroneous, since the points Q, R should be alike privileged to traverse every region of infinite space to which the point P has been admitted. As the case stands, they are mutually located within distances which, however great, are exclusively supposed to be finite. That is, to put the matter briefly, Professor De Morgan, in a most exceptional manner, assigns a range to the third point infinitely greater than that which is virtually accorded by him to the points Q and R. These defects necessarily lead to wrong conclusions.



Professor De Morgan next advances certain views with respect to the "four point problem," and arrives at an even chance between a convex and a re-entering quadrilateral. The solution given by him is as follows:—

"This question about the distinction between space with infinitely distant boundary and infinite space, will always raise discussion. Any triangle, *however great*,—and therefore an infinite triangle,—has six external spaces, each infinitely greater than the triangle. Three, say P, Q, R, are the angular spaces of the opposite angles; and, S being the triangle, the other three are angular spaces with the triangle cut off, or $P-S$, $Q-S$, $R-S$. If four points be taken quite at hazard, all points of the plane being equally likely, all triangles are equally likely to be formed by three of the points. Any one triangle being taken, the chance that the fourth point so falls that one of the four shall be in the triangle of the other three is to the chance against it as $P+Q+R+S$ to $P+Q+R-3S$; that is, S being infinitely small compared with P, or Q, or R, the chance is an even chance. Hence, by repeating all the equally probable triangles on common principles, it is an even chance, four points being taken at hazard, that one shall be in the triangle of the other three. I am aware of a very ingenious proposal of solution which gives 1 to 3 instead of 1 to 1, and I am prepared to discuss it when it shall be published. I will not at present pronounce decidedly for the above solution; but I am myself utterly unable to see how it can be questioned."



It is here obviously true of a *finite* triangle, *however great*, that it must have the six external spaces as exhibited in the annexed diagram. But again, exception must be taken to the validity of the proposition which asserts that an *absolutely infinite* triangle can have such adjuncts. Any conception of such external spaces, or indeed of the sides of the triangle, must at once put a direct negative upon its presumed infinite character, which is tantamount to a positive contradiction.

In order that the aggregate combinations may be complete, it is clear that all the four points should alike pass through every possible change of position, and therefore that the vertices or corners of the assumed triangle should severally and jointly transverse every region of space, which the fourth point has been supposed to occupy. It is manifestly wrong, therefore, to suppose that the triangle S is in all cases infinitely small compared with P, Q, or R; and the probability deduced from this supposition must be erroneous. Professor De Morgan is instinctively right in his hesitation to pronounce decidedly in favour of this method.

The substance of a true solution for the general case in which the four points are supposed to be taken at random within any given enclosed area may be sketched thus. Let P, Q, R be three of the points; then, as the fourth point S may be posited anywhere within the proposed area, it is evident that the probability that it shall fall within the triangle PQR is determined by the fraction $\dfrac{\text{triangle PQR}}{\text{given area}}$. Therefore, when the points occupy all positions on the given area, we shall have

$$\begin{aligned} \frac{\text{average triangle}}{\text{given area}} &= \text{prob. of S within PQR} \\ &= \quad ,, \quad \text{P} \quad ,, \quad \text{QRS} \\ &= \quad ,, \quad \text{Q} \quad ,, \quad \text{RSP} \\ &= \quad ,, \quad \text{R} \quad ,, \quad \text{SPQ.} \end{aligned}$$

These four separate probabilities are by symmetry precisely identical in value, since the positions of the four points admit of being mutually interchanged or permuted. The several conditions are also individually exclusive, and the sum of the four equal values obviously makes up the complete probability that the quadrilateral shall be re-entering. This last probability is therefore $\dfrac{\text{4 times average triangle}}{\text{given area}}$; that is, the required probability of a re-entering quadrilateral is found by comparing four times the average area of all inserted triangles with the given area.

If the proposed area be triangular, Mr. Stephen Watson and Professor Sylvester have shown (Quest. 1229, *Reprint*, Vol. II. p. 95, and Vol. IV. p. 101) that the average inserted triangle is $\frac{1}{12}$th of the given area. Hence, when the four points are to be taken on the surface of any given triangle, the probability of a re-entering quadrilateral is $\frac{1}{3}$.

1503. (Proposed by Professor SYLVESTER.)—A table has n holes bored in its rim, into which ν pegs are to be inserted at random, ν being not greater than $\frac{1}{2}n$. Show that the probability of there being no two pegs without one or more unoccupied holes between them will be equal to $\frac{\pi(n-\nu)\,.\,\pi(n-\nu-1)}{\pi(n-1)\,.\,\pi(n-2\nu)}$, and, if ν is given, approaches to certainty as n becomes indefinitely great.

[N.B.—$\pi(n)$ here denotes the product $1.2.3....n$.]

I. *Solution by* SAMUEL ROBERTS, M.A.

We may suppose the same peg to make the circuit of the table, both when we seek the possible and when we seek the favourable cases. Hence the ratio of these numbers will be the same as that of the corresponding numbers when one hole is occupied by a peg.

On this supposition the number of possible cases is the number of variations of $n-1$ things taken $\nu-1$ together, or $\frac{\pi(n-1)}{\pi(n-\nu)}$. Let a circle be drawn with $n-\nu$ radii which represent the $n-\nu$ holes, which necessarily remain unoccupied. We may consider these radii as fixed, while the $n-\nu$ spaces between them are possible favourable positions of the pegs. One of these spaces being occupied, the corresponding number of favourable cases is the number of variations of $n-\nu-1$ things taken $\nu-1$ together, or $\frac{\pi(n-\nu-1)}{\pi(n-2\nu)}$, and the ratio of these values is $\frac{\pi(n-\nu)}{\pi(n-1)}\,.\,\frac{\pi(n-\nu-1)}{\pi(n-2\nu)}$.

When the expression is put in the form of a fraction, having its numerator and denominator of the same degree in n, we see that as n becomes great, the value approaches unity, if ν remains constant.

II. *Solution by* G. C. DE MORGAN, M.A.

Let $z_{n,\,\nu}$ be the number of ways in which ν pegs may be inserted in n holes, in accordance with the conditions of the problem, on the supposition that the holes are bored in a row of which the first and last are not to be considered as adjacent. To get the number of ways in which they may be inserted when the holes are bored completely round the table, we must subtract the number of ways in which the first and last are both filled up, or $z_{n-4,\,\nu-2}$.

Now $z_{n,\,\nu}$ is made up of the number of ways in which the first hole is filled, together with the number of ways in which it is not filled; hence

$$z_{n,\,\nu} = z_{n-2,\,\nu-1} + z_{n-1,\,\nu}.$$

Let $z_{\nu,\,n}$ be the coefficient of $x^n y^\nu$ in $\phi(x, y)$: then we have

$$\phi(x,y) = z_{0,0} + z_{1,0} \cdot x + z_{2,0} \cdot x^2 + \dots + z_{1,1} \cdot xy + z_{2,1} \cdot x^2y + \dots$$

$$x \cdot \phi(x,y) = z_{0,0} \cdot x + z_{1,0} \cdot x^2 + \dots\dots\dots\dots + z_{1,1} \cdot x^2y + \dots$$

$$x^2y \cdot \phi(x,y) = z_{0,0} \cdot x^2y + z_{1,0} \cdot x^3y + \dots\dots + z_{1,1} \cdot x^3y^2 + \dots$$

From these and the equation of differences above, remembering that $z_{0,0} = 1$, $z_{1,0} = 1$, &c., $z_{1,1} = 1$, we get $\left\{1 - x - x^2y\right\} \cdot \phi(x,y) = 1 + xy$,

therefore $\phi(x,y) = \dfrac{1+xy}{1-x-x^2y} = 1 + xy + x \cdot (1+xy)^2 + x^2 \cdot (1+xy)^3 + \dots$

The term obtained from $x^{k-1} \cdot (1+xy)^k$ which has y^ν as a factor, is $\dfrac{k \cdot (k-1) \dots (k-\nu+1)}{1 \cdot 2 \dots \nu} \cdot x^{\nu+k-1} \cdot y^\nu$, and we must take k in such a manner that $\nu + k - 1 = n$, or $k = n - \nu + 1$. This gives for the coefficient of $x^n y^\nu$, $\dfrac{(n-\nu+1) \cdot (n-\nu) \dots (n-2\nu+2)}{1 \cdot 2 \dots \nu}$, or, using Professor SYLVESTER'S notation, $\dfrac{\pi \cdot (n-\nu+1)}{\pi(\nu) \cdot \pi(n-2\nu+1)} = z_{n,\nu}$.

Hence $z_{n,\nu} - z_{n-4,\nu-2} = \dfrac{\pi(n-\nu+1)}{\pi(\nu) \cdot \pi(n-2\nu+1)} - \dfrac{\pi(n-\nu-1)}{\pi(\nu-2) \cdot \pi(n-2\nu+1)}$

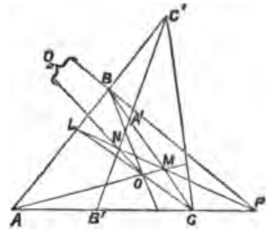
$$= n \cdot \frac{\pi \cdot (n-\nu-1)}{\pi(\nu) \cdot \pi(n-2\nu)}.$$

Dividing by $\dfrac{\pi(n)}{\pi(\nu) \cdot \pi(n-\nu)}$, the total number of ways in which ν pegs may be inserted in n holes, we get $\dfrac{\pi(n-\nu) \cdot \pi(n-\nu-1)}{\pi(n-1) \cdot \pi(n-2\nu)}$, the probability required. The limit of this, when n is infinite, is given by substituting $\sqrt{(2\pi)} \cdot (n-\nu)^{n-\nu+\frac{1}{2}} \cdot \epsilon^{-(n-\nu)}$ for $\pi(n-\nu)$, &c. This gives, after reduction,

$\dfrac{\left(1-\frac{\nu}{n}\right)^{n-\nu+\frac{1}{2}} \cdot \left(1-\frac{\nu+1}{n}\right)^{n-\nu-\frac{1}{2}}}{\left(1-\frac{1}{n}\right)^{n-\frac{1}{2}} \cdot \left(1-\frac{2\nu}{n}\right)^{n-2\nu+\frac{1}{2}}}$, the limit of which for a given value of ν is $\dfrac{\epsilon^{-\nu} \cdot \epsilon^{-(\nu+1)}}{\epsilon^{-1} \cdot \epsilon^{-2\nu}}$, or 1.

1799. (Proposed by Professor WHITWORTH, M.A.)—Three conics are described so that each of them passes through the same point O, and through the extremities of two of the diagonals of the same complete quadrilateral. Prove that if O_1, O_2, O_3 are their other points of intersection, then OO_1, OO_2, OO_3 are the tangents to the three conics at O.

I. *Solution by* J. Dale; H. Tomlinson; *and others.*



Let AA', BB', CC' be the diagonals of the quadrilateral; A', B', C' being collinear vertices; and O the common point. Draw the lines AOM, cutting BC in M; COL, cutting AB in L; and LNMP, cutting B'C' in N and AC in P; also join ON, PB, and produce these lines to meet in O_2. In the hexagon $AA'BB'OO_2$ the lines OA, BA'; AB', O_2B; B'A', O_2O intersect in the collinear points M, P, N; therefore, by Pascal's theorem, O_2 lies on the conic AA'BB'O. In the hexagon $BB'CC'OO_2$ the lines OC, C'B; CB', BO_2; B'C', O_2O; intersect in the collinear points L, P, M; therefore O_2 lies on the conic BB'CC'O. In the pentagon CC'AA'O, the straight line OO_2 is drawn through O so that the lines OC, C'A; CA', AO; A'C', OO_2 intersect in the collinear points L, M, N; therefore OO_2 is a tangent to the conic CC'AA'O. In the other two cases the proof is similar.

II. *Solution by the* Proposer; W. H. Laverty; E. McCormick; W. Chadwick; *and others.*

Let AA', BB', CC' be the diagonals of the quadrilateral; and, taking the four sides BC, CA, AB, A'B'C' as lines of reference for quadrilinear coordinates, let $(\alpha', \beta', \gamma', \delta')$ be the coordinates of O. Then we may write the equations to the conics which pass through $OO_2O_3BB'CC'$, $OO_3O_1CC'AA'$, $OO_1O_2AA'BB'$, respectively

$$S_1 \equiv \frac{\alpha\delta}{\alpha'\delta'} - \frac{\beta\gamma}{\beta'\gamma'} = 0,\quad S_2 \equiv \frac{\alpha\beta}{\alpha'\beta'} - \frac{\gamma\delta}{\gamma'\delta'} = 0,\quad S_3 \equiv \frac{\alpha\gamma}{\alpha'\gamma'} - \frac{\beta\delta}{\beta'\delta'} = 0.$$

Now $S_2 - S_3 \equiv \frac{\alpha}{\alpha'}\left(\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'}\right) - \frac{\delta}{\delta'}\left(\frac{\gamma}{\gamma'} - \frac{\beta}{\beta'}\right) \equiv \left(\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'}\right)\left(\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'}\right)$;

hence $\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} = 0$ and $\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} = 0$ are the equations to a pair of common chords of S_2 and S_3. But $\frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} = 0$ is the equation to the chord OA; therefore $\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} = 0$ must represent the chord O_1A'.

Similarly we can show that

$$\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0 \text{ must represent the chord } O_1A.$$

Hence the point O_1 is given by the equations

$\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} = 0$ and $\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0$; and it therefore lies upon the line whose equation is

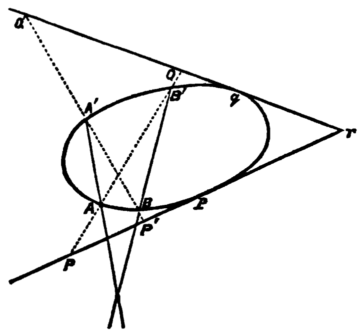
$$\frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} - \left(\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'}\right) = 0, \text{ or } \frac{\alpha}{\alpha'} - \frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} + \frac{\delta}{\delta'} = 0.$$

But this is known to be the equation to the tangent at $(\alpha', \beta', \gamma', \delta')$, or O, to the conic S_1; that is to say, OO_1 is the tangent at O to S_1.

Similarly OO_2 and OO_3 are the tangents at O to the other conics.

1941. (Proposed by F. D. THOMSON, M.A.)—AA′B′B is a quadrilateral inscribed in a conic. Two tangents PP′, QQ′ meet the diagonals AB′, A′B in the points P, P′, Q, Q′ respectively. Show that a conic can be described so as to touch AA′, BB′, and also pass through the four points P, P′, Q, Q′.

Solution by the PROPOSER.



Let $LM = R^2$ be the equation to the conic, referred to the two tangents and their chord of contact. Let a, b, a', b', be the quantities defining as usual the points A, B, A′, B′. Then the equation to a conic round PP′QQ′ is of the form

$$[AB'][A'B] = \lambda [PP'][QQ'] \text{ (i).}$$

Now the equations to AB′, A′B are, respectively,

$$ab'L-(a+b')R+M=0\,..\,[AB'],$$
$$a'bL-(a'+b)R+M=0\,..\,[A'B];$$

therefore (i.) becomes

$$\{ab'L-(a+b')R+M\}\{a'bL-(a'+b)R+M\} = \lambda LM \quad\ldots\ldots\ldots \text{(ii).}$$

The equation to AA′ is $\quad aa'L-(a+a')R+M=0 \ldots\ldots\ldots\ldots\ldots$ (iii);
therefore to find when (iii.) meets (ii.), we have

$$\{(ab'-aa')L+(a'-b')R\}\{(a'b-aa')L+(a-b)R\} = \lambda L\{(a+a')R-aa'L\},$$

or, $$(b'-a')(b-a)(aL-R)(a'L-R) = \lambda L\{(a+a')R-aa'L\}$$

or, $$aa'L^2\{(b'-a')(b-a)+\lambda\} - LR(a+a')\{(b'-a')(b-a)+\lambda\} + (b'-a')(b-a)R^2 = 0 \ldots\ldots\ldots \text{(iv.),}$$

therefore if AA′ touch (ii.), we must have

$$(a+a')^2\{(b'-a')(b-a)+\lambda\}^2 = 4aa'\{(b'-a')(b-a)+\lambda\}(b'-a')(b-a),$$

therefore $$(b'-a')(b-a)+\lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v.)}$$

or, $$(a+a')\{(b'-a')(b-a)+\lambda\} = 4aa'(b'-a')(b-a).$$

The value of λ given by (v.) is unchanged if we write b, b' for a, a', and therefore if (ii.) touches AA′ it also touches BB′.

COR. 1.—If we substitute for λ in (ii.) from (v.) the equation to the conic becomes

$$\{ab'L-(a+b')R+M\}\{a'bL-(a'+b)R+M\} + (b'-a')(b-a)LM = 0,$$

or, $$aa'bb'L^2 - \{ab'(a'+b)+a'b(a+b')\}LR + (aa'+bb')LM + M^2 - (a+a'+b+b')MR + (a+b')(a'+b)R^2 = 0,$$

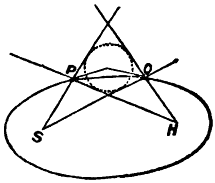
or, $$\{aa'L-(a+a')R+M\}\{bb'L-(b+b')R+M\} + (a-b)(a'-b')R^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vi.),}$$

or, $$[AA'][BB'] + (a-b)(a'-b')[pq]^2 = 0.$$

Hence pq is the chord of contact of the tangents AA′, BB′.

COR. 2.—Precisely the same work in *tangential* coordinates would lead to the reciprocal of this theorem, viz.: If ABCD be a quadrilateral circumscribing a conic, and P, Q any two points on the conic, then a conic can be described touching AP, AQ, CP, CQ and passing through the points B and D. Also the pole of PQ with respect to the first conic will be the pole of BD with respect to the second conic.

A particular case of the theorem is when the points B and D are the circular points at infinity, and therefore A and C the foci; and then we get the following theorem (noticed by Mr. C. TAYLOR in the *Messenger of Mathematics*):—that if P and Q be any points on a conic, S and H the foci, a *circle* can be inscribed in the quadrilateral formed by SP, SQ, HP, HQ, having its *centre* at the pole of PQ with respect to the conic.



1949. (Proposed by Professor CAYLEY.)—Find the conic of five-pointic intersection at any point of the cuspidal cubic $y^3 = x^2z$.

I. *Solution by* W. H. H. HUDSON, M.A.

Let P be the proposed point. Since a conic and a cubic intersect in 6 points, let Q be the other point of intersection. Let the tangent at P meet the cubic again in p, and let the tangent at p meet the cubic again in p', then shall PQp' be a straight line. For, if p'' be the point in which PQ meets the cubic again, the cubic made up of Pp taken twice and PQ intersect the given cubic in the 9 points $P^5 . p^2 . Q . p''$. Also the cubic made up of the conic and pp' meets the given cubic in the 9 points $P^5 . p^2 . Q . p'$; and 8 of these $P^5 . p^2 . Q$ being coincident, it follows, by a well-known theorem, that the 9th is so likewise, or p', p'' are the same point.

Let now $S=0$ be the equation of the conic; $u=0$, $v=0$, $w=0$ of the lines Pp, PQ, pp' respectively; then we must have

$$y^3 - x^2z = Sw - \lambda u^2 v.$$

We can form the equations $u=0$, $v=0$, $w=0$; then, equating coefficients, we have 10 equations to determine the 6 coefficients of $S=0$ and λ. Solving 7 of these equations, the values found satisfy the other three: and the result thus obtained is

$$5\left(\frac{x}{h}\right)^2 + 45\left(\frac{y}{k}\right)^2 - \left(\frac{z}{l}\right)^2 + 15\frac{yz}{kl} - 40\frac{zx}{lh} - 24\frac{xy}{hk} = 0,$$

h, k, l being the coordinates of the point of five-pointic intersection.

II. *Solution by the* PROPOSER.

The equation $y^3 = x^2z$, is satisfied by the values $x : y : z = 1 : \theta : \theta^3$; and conversely, to any given value of the parameter θ there corresponds a point

on the cubic $y^3 = x^2 z$. Consider the five points corresponding to the values $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ respectively; the equation of the conic through these five points is

$$\begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ 1\ , & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0,$$

where the remaining four lines of the determinant are obtained from the second line by writing therein $\theta_2, \theta_3, \theta_4, \theta_5$ successively in place of θ_1. Writing for shortness $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ to denote the product of the differences of the quantities $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, the equation contains the factor $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, and we may therefore write it in the simplified form

$$\frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} x^2, & y^2\ , & z^2\ , & yz, & zx, & xy \\ 1\ , & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0.$$

Hence putting in this equation $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \phi$, we have the equation of the conic of five pointic intersection at the point (ϕ). The result in its reduced form may be obtained directly without much difficulty, but it is obtained most easily as follows: let the function on the left-hand of the foregoing equation be represented by

$$\left\{a, b, c, f, g, h\right\} (x, y, z)^{2*}$$

then writing $x : y : z = 1 : \theta : \theta^3$, we have

$(a, b, c, f, g, h)\ (1, \theta, \theta^3)^2$

$$= \frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2\ , & \theta^6\ , & \theta^4\ , & \theta^3\ , & \theta \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix}$$

$$= \frac{(\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4)(\theta-\theta_5)}{\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2, & \theta^6, & \theta^4, & \theta^3, & \theta \\ 1, & \theta^2_1, & \theta^6_1, & \theta^4_1, & \theta^3_1, & \theta_1 \\ \vdots & & & & & \end{vmatrix}$$

$$= (\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4)(\theta-\theta_5)(\theta+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5);$$

[for the determinant, which is a function of the order 16 in the quantities $(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ conjointly, divides by $\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, which is a function of the order 15; and as the quotient is a symmetrical function of θ $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, it must be equal, save to a numerical factor which may be disregarded, to $\theta+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5$].

Hence if ϕ be the parameter of the given point, writing $\theta_1 = \theta_2 = \theta_3 = {}_4 = \theta_5 = \phi$, we have

$$\left\{a, b, c, f, g, h\right\} (1, \theta, \theta^3)^2 = (\theta-\phi)^5 (\theta+5\phi)$$
$$= \left\{1, 0, -15, +40, -45, +24, -5\right\} (\theta, \phi)^6,$$

where the left-hand side is

$a + b\theta^2 + c\theta^6 + f\theta^4 + g\theta^3 + h\theta, = \left\{c, 0, f, g, b, h, a\right\} (\theta, 1)^6,$

that is, we have

$c = 1, f = -15\phi^2, g = 40\phi^3, b = -45\phi^4, h = 24\phi^5, a = -5\phi^6,$

and the equation of the conic of five-pointic intersection therefore is

$$\left\{-5\phi^6, -45\phi^4, 1, -15\phi^2, 40\phi^3, 24\phi^5\right\} (x, y, z)^2 = 0,$$

* I write $\left\{\ \right\}$, instead of the usual arrow-headed parenthesis, to signify the omission of the binomial coefficients, viz., $\left\{a, b, c, f, g, h\right\}(x, y, z)^2$ means $ax^2+by^2+cz^2+fyz+gzx+hxy$.

or, what is the same thing,

$$-5\phi^6x^2-45\phi^4y^2+z^2-15\phi^2\,yz+40\phi^3\,zx+24\phi^5\,xy=0,$$

which is the required result.

NOTE.—The condition in order that any six points $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ of the cubic $y^3=x^2z$ may lie on a conic, is

$$\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6=0.$$

1958. (Proposed by J. GRIFFITHS, M.A.)—Let P denote a point in the plane of a given triangle ABC; α, β, γ the feet of the perpendiculars drawn from P upon the sides BC, CA, AB. Then if the triangle $\alpha\beta\gamma$ is homologous with the given one ABC, the locus of P is a cubic which passes through (1) the angular points of the triangle; (2) the centres of the inscribed and escribed circles; (3) the point of intersection of the three perpendiculars; (4) the centre, O, of the circumscribing circle; (5) the points L, M, N, where the radii AO, BO, CO produced meet this circle again. Moreover, if P′ denote the inverse of P with respect to the sides of the given triangle, show that P′ also lies on the cubic-locus in question.

I. *Solution by* J. DALE; *and others.*

Let x, y, z be the coordinates of P, the equations of Aα, Bβ, Cγ will be

$$\frac{\beta}{y+x\cos C}=\frac{\gamma}{z+x\cos B},\quad \frac{\gamma}{z+y\cos A}=\frac{\alpha}{x+y\cos C},$$

$$\frac{\alpha}{x+z\cos B}=\frac{\beta}{y+z\cos A};$$

and the condition that these three lines should meet in a point is

$$(y+z\cos A)(z+x\cos B)(x+y\cos C)=(y\cos A+z)(z\cos B+x)(x\cos C+y)\;\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

or, $$(\cos A-\cos B\cos C)\,x\,(y^2-z^2)+(\cos B-\cos C\cos A)\,y\,(z^2-x^2)$$
$$+(\cos C-\cos A\cos B)\,z\,(x^2-y^2)=0\ldots\ldots\ldots\ldots\text{(ii.)}$$

(1.) This cubic passes through the angles of the triangle of reference, the coordinates of which are $(y=0,\ z=0)$, $(z=0, x=0)$, $(x=0, y=0)$.

(2.) It is satisfied by the values $x^2=y^2=z^2$, or

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1},\quad \frac{x}{-1}=\frac{y}{1}=\frac{z}{1},\quad \frac{x}{1}=\frac{y}{-1}=\frac{z}{1},\quad \frac{x}{1}=\frac{y}{1}=\frac{z}{-1},$$

which represent the centres of the inscribed and escribed circles.

(3.) It is also satisfied by $x\cos A=y\cos B=z\cos C$, which represents the intersection of the perpendiculars.

(4.) The centre O of the circumscribing circle $\dfrac{x}{\cos A}=\dfrac{y}{\cos B}=\dfrac{z}{\cos C}$ lies on the cubic.

(5.) The points L, M, N, where the radii AO, BO, CO produced meet the circumscribing circle again, are given by the equations

$$\frac{x}{-\cos B \cos C} = \frac{y}{\cos B} = \frac{z}{\cos C}, \quad \frac{x}{\cos A} = \frac{y}{-\cos C \cos A} = \frac{z}{\cos C},$$

$$\frac{x}{\cos A} = \frac{y}{\cos B} = \frac{z}{-\cos A \cos B};$$

and these values also satisfy the equation of the cubic.

for x, y, z; hence if any point P lie on the cubic, its inverse will also lie on

(6.) The equation remains unchanged when $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ are substituted for the cubic; (3) and (4) are examples of this.

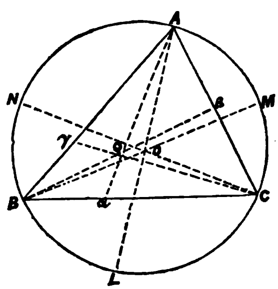
(7.) The cubic is central, having O for its centre; (1) and (5) are examples of this.

[Mr. Greer remarks that the curve intersects each perpendicular of the triangle at infinity, three points which, *at first sight*, do not appear to satisfy the locus according to the geometrical condition by which it is defined. If the triangle ABC be turned round the centre of the circumscribing circle through two right angles, carrying the cubic with it, the curve returns into coincidence with its original position. Thus it is symmetrical all round this centre, which, therefore, must be one of its points of inflexion. Verifying this algebraically, there is brought to light the following identical relation, subsisting amongst the cosines of the angles of a plane triangle, viz., (writing A for cos A, &c.)

$$\begin{vmatrix} B^2-C^2, & C(B^2-A^2), & B(A^2-C^2) \\ C(B^2-A^2), & C^2-A^2, & A(C^2-B^2) \\ B(A^2-C^2), & A(C^2-B^2), & A^2-B^2 \end{vmatrix} = 0].$$

II. *Solution by* F. D. Thomson, M.A.

Let Q be the point $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ (trilinear coordinates). Draw AQα, BQβ, CQγ to meet the sides, and let the point Q be such that the perpendiculars at α, β, γ meet in a point P. We have to find the locus of P.



The equation to the perpendicular to BC through α must be of the form $kx + ny - mz = 0$, where k is some constant, and since the line is perpendicular to $x = 0$, we have $k + m \cos B - n \cos C = 0$, therefore equation to perpendicular at α is

$$(n \cos C - m \cos B)\, x + ny - mz = 0,$$

or, $\quad m(x \cos B + z) = n(x \cos C + y)$, or $m\,[BN] = n\,[CM]$ ...... (iii.),

if L, M, N be the points on the circumscribing circle diametrically opposite to A, B, C, and [BN] denote the equation to BN.

Similarly equation to perpendicular at β is $n\,[CL] = l\,[AN]$ ...... (iv.),

and equation to perpendicular at γ is $l\,[AM] = m\,[BL]$ ...... (v.);

therefore, eliminating l, m, n from (iii.), (iv.), (v.), we have

$$[BN]\,[CL]\,[AM] = [CM]\,[AN]\,[BL] \; \ldots\ldots\ldots\ldots (vi.),$$

the equation to a cubic through the intersections of BN with CM, AN, BL, that is to say, through ∞, N, B,
the equation to a cubic through the intersections of CL with CM, AN, BL, that is to say, through C, ∞, L,
the equation to a cubic through the intersections of AM with CM, AN, BL, that is to say, through M, A, ∞.
Therefore the cubic passes through A, B, C, L, M, N, and has its asymptotes perpendicular to the sides of the triangle.

Writing the equation in full, it will reduce to the form (ii.) of the first solution, which is seen to be satisfied by the points in (2), (3), (4); also by

$$\frac{x}{\cos A-\cos B\cos C}=\frac{y}{\cos B-\cos C\cos A}=\frac{z}{\cos C-\cos A\cos B}\;\ldots\ldots\ldots(S),$$

and the points where AS, BS, CS meet the sides BC, CA, AB.

It is seen from the form of the equation that if x, y, z satisfy it, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ will also satisfy it. Hence the point

$$x(\cos A-\cos B\cos C)=y(\cos B-\cos C\cos A)=z(\cos C-\cos A\cos B)\ldots(S')$$

is also on the curve.

It may be shown that the tangents at A, B, C, S meet in S′, or with the definitions of a paper in the *Messenger of Mathematics*, (Vol. III., p. 15), that S′ is the *satellite* of each of the four parts A, B, C, S.

It may be shown, as in the paper referred to, that S is the satellite of each of the centres of the in- and e-scribed circles, and many similar properties may be deduced.

The above investigation suggests the question as to what is the locus of the point Q in the figure.

We have from the foregoing, if l, m, n be proportional to the coordinates of Q,

$$(n\cos C-m\cos B)\,x+ny-mz=0,$$

and two similar equations.

Hence, eliminating x, y, z, we get a cubic which reduces to

$$l\cos A\,(m^2\sin^2 B-n^2\sin^2 C)+m\cos B\,(n^2\sin^2 C-l^2\sin^2 A)+ n\cos C\,(l^2\sin^2 A-m^2\sin^2 B)=0,$$

or the locus of Q is the cubic

$$x\cos A\,(b^2y^2-c^2z^2)+y\cos B\,(c^2z^2-a^2x^2)+z\cos c\,(a^2x^2-b^2y^2)=0,$$

which may easily be seen to pass through A, B, C, the centre of gravity G, the intersection of perpendiculars, and A′, B′, C′, the angular points of the triangle formed by drawing lines through A, B, C parallel to the opposite sides.

The tangents at A, B, C, meet on the curve in T the intersection of perpendiculars.

The tangents at A′, B′, C′, G meet in a point on the curve.

1693. (Proposed by Dr. BOOTH, F.R.S.)—Let $\Pi(m,\omega)$, $\Pi(m,\phi)$, $\Pi(m,\psi)$ be three arcs of a parabola measured from the vertex; ω, ϕ, and ψ their amplitudes (that is, the inclinations to the axis of the focal perpen-

diculars on the tangents at the extremities of the arcs, being connected by the equation $\tan\omega = \tan\phi\sec\psi + \sec\phi\tan\psi$. Show that their algebraic sum is equal to the product of the ordinates at their extremities divided by the square of the semi-parameter; and apply the theorem to the particular case in which $\tan\omega = 2$, $\tan\phi = \frac{1}{2}$, $\tan\psi = \frac{1}{2}\sqrt{5}$.

1729. (Proposed by Dr. BOOTH, F.R.S.)—Prove that the difference between any parabolic arc (measured from the vertex) and its ordinate is equal to the arc cut off by a focal chord, the amplitudes being connected by the condition $\tan\omega = \sec\phi + \operatorname{cosec}\phi$.

I. *Solution by* J. DALE; E. FITZGERALD; E. McCORMICK; *and others.*

(**1693.**) Putting $4m$ for the parameter of the parabola, and r for the radius vector of an arc whose amplitude is θ, we have $r = m\sec^2\theta$,

$$\therefore\ ds = \left\{(dr)^2 + (rd\,.\,2\theta)^2\right\}^{\frac{1}{2}} = 2m\,(1+\tan^2\theta)^{\frac{1}{2}}\,d\,.\tan\theta;$$

$$\therefore\ \Pi\,(m,\omega) = m\left\{\frac{\sin\omega}{\cos^2\omega} + \log\left(\frac{1+\sin\omega}{\cos\omega}\right)\right\},\ \&c.;$$

$$\therefore\ \Pi\,(m,\omega) - \Pi\,(m,\phi) - \Pi\,(m,\psi) = m\left\{\frac{\sin\omega}{\cos^2\omega} - \left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\psi}{\cos^2\psi}\right)\right\}$$

$$+\, m\log\left\{\frac{1+\sin\omega}{\cos\omega} \div \left(\frac{1+\sin\phi}{\cos\phi}\right).\left(\frac{1+\sin\psi}{\cos\psi}\right)\right\}.$$

But from the equation of condition we have

$$\frac{1}{\cos\omega} = \frac{1+\sin\phi\sin\psi}{\cos\phi\cos\psi},\quad \frac{1+\sin\omega}{\cos\omega} = \left(\frac{1+\sin\phi}{\cos\phi}\,.\,\frac{1+\sin\psi}{\cos\psi}\right);$$

$$\therefore\ \log\left\{\frac{1+\sin\omega}{\cos\omega} \div \left(\frac{1+\sin\phi}{\cos\phi}\,.\,\frac{1+\sin\psi}{\cos\psi}\right)\right\} = \log(1) = 0.$$

We have also $$\frac{\sin\omega}{\cos^2\omega} = \frac{(\sin\phi+\sin\psi)\,(1+\sin\phi\sin\psi)}{\cos^2\phi\cos^2\psi},$$

and $$\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\psi}{\cos^2\psi} = \frac{(\sin\phi+\sin\psi)\,(1-\sin\phi\sin\psi)}{\cos^2\phi\cos^2\psi};$$

$$\therefore\ \frac{\sin\omega}{\cos^2\omega} - \left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\psi}{\cos^2\psi}\right) = \frac{2\,(\sin\phi+\sin\psi)\sin\phi\sin\psi}{\cos^2\phi\cos^2\psi}$$

$$= 2\tan\omega\tan\phi\tan\psi.$$

Moreover $y_\omega = 2m\tan\omega$, &c.; $\therefore\ y_\omega y_\phi y_\psi = 8m^3\tan\omega\tan\phi\tan\psi$;

therefore $\Pi\,(m,\omega) - \Pi\,(m,\phi) - \Pi\,(m,\psi) = y_\omega y_\phi y_\psi \div (2m)^2$.

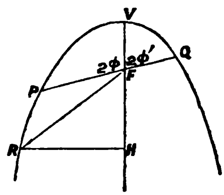
In the particular case given, the equation of condition is satisfied; and the algebraical sum is $m\sqrt{5}$.

(**1729.**) This may be obtained from the foregoing solution by supposing $\psi = \frac{1}{2}\pi - \phi$, when we have

$\Pi\,(m,\omega) - \Pi\,(m,\phi) - \Pi\,(m,\psi) = 2m\tan\omega = y_\omega$, or

$\Pi\,(m,\omega) - y_\omega = \Pi\,(m,\phi) + \Pi\,(m,\frac{1}{2}\pi - \phi) =$ arc of focal chord of amplitude ϕ.

II. *Solution by* W. S. B. Woolhouse



Let VP be any arc of a parabola, vertex V, focus F, and equation $y^2 = 4mx$. Then if 2ϕ denote the polar angle VFP, we shall have

$$x = m\tan^2\phi,\quad y = 2m\tan\phi;$$

and
$$s_\phi = \sqrt{x(m+x)} + m\log\frac{\sqrt{x}+\sqrt{(m+x)}}{\sqrt{m}}$$
$$= m\left(\frac{\sin\phi}{\cos^2\phi} + \log\frac{1+\sin\phi}{\cos\phi}\right)$$
$$= m\left(\frac{\sin\phi}{\cos^2\phi} + \frac{1}{2}\log\frac{1+\sin\phi}{1-\sin\phi}\right) \quad\ldots\ldots(1).$$

Therefore, if $2\phi'$ be the polar angle of any second arc, the sum of the two arcs will be

$$s_\phi + s_{\phi'} = m\left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\phi'}{\cos^2\phi'}\right) + \frac{m}{2}\log\frac{(1+\sin\phi)(1+\sin\phi')}{(1-\sin\phi)(1-\sin\phi')} \quad\ldots\ldots(2).$$

Let us compare this value with the length of another parabolic arc having 2ω as polar angle, viz.,

$$s_\omega = m\frac{\sin\omega}{\cos^2\omega} + \frac{m}{2}\log\frac{1+\sin\omega}{1-\sin\omega} \quad\ldots\ldots(3).$$

Suppose the transcendental or final terms of (2) and (3) to be equal; then

$$\frac{1+\sin\omega}{1-\sin\omega} = \frac{(1+\sin\phi)(1+\sin\phi')}{(1-\sin\phi)(1-\sin\phi')} \quad\ldots\ldots(4),$$

which gives

$$\sin\omega = \frac{\sin\phi+\sin\phi'}{1+\sin\phi\sin\phi'},\quad \cos\omega = \frac{\cos\phi\cos\phi'}{1+\sin\phi\sin\phi'},\quad \tan\omega = \frac{\sin\phi+\sin\phi'}{\cos\phi\cos\phi'} \quad\ldots(\alpha);$$

also,
$$\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\phi'}{\cos^2\phi'} = \frac{\sin\phi\,(1-\sin^2\phi') + \sin\phi'\,(1-\sin^2\phi)}{\cos^2\phi\cos^2\phi'}$$
$$= \frac{(\sin\phi+\sin\phi')(1-\sin\phi\sin\phi')}{\cos^2\phi\cos^2\phi'} = \tan\omega\,\frac{1-\sin\phi\sin\phi'}{\cos\phi\cos\phi'}$$

and
$$\frac{\sin\omega}{\cos^2\omega} = \frac{\tan\omega}{\cos\omega} = \tan\omega\,\frac{1+\sin\phi\sin\phi'}{\cos\phi\cos\phi'};$$

$$\therefore\quad \frac{\sin\omega}{\cos^2\omega} - \left(\frac{\sin\phi}{\cos^2\phi} + \frac{\sin\phi'}{\cos^2\phi'}\right) = 2\tan\omega\tan\phi\tan\phi' \quad\ldots\ldots(5).$$

Hence, by virtue of (4) and (5), we find that

$$\left.\begin{aligned} &s_\omega - (s_\phi + s_{\phi'}) = 2m\tan\omega\tan\phi\tan\phi' \\ \text{or,}\quad &s_\phi + s_{\phi'} = s_\omega - y_\omega\tan\phi\tan\phi' \end{aligned}\right\} \quad\ldots\ldots(\beta).$$

The formulæ (α), (β) enable us to evaluate the sum of any two arcs of a parabola by means of a single arc.

In the same manner may be determined analogous formulæ which will enable us to evaluate the difference of any two arcs of a parabola by means of a single arc. We need, however, only to change the algebraic sign of ϕ',

since the formula (2) will obviously then express $s_\phi - s_{\phi'}$. Thus putting $-\phi'$ for ϕ, the relations (α) and (β) become

$$\left.\sin\omega' = \frac{\sin\phi - \sin\phi'}{1-\sin\phi\sin\phi'},\ \cos\omega' = \frac{\cos\phi\cos\phi'}{1-\sin\phi\sin\phi'},\ \tan\omega' = \frac{\sin\phi-\sin\phi'}{\cos\phi\cos\phi'}\right\}\ \ldots(\gamma)$$

$$\left.\begin{aligned}&(s_\phi - s_{\phi'}) - s_{\omega'} = 2m\tan\omega'\tan\phi\tan\phi'\\ \text{or,}\quad &s_\phi - s_{\phi'} = s_{\omega'} + y_{\omega'}\tan\phi\tan\phi'\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(\delta).$$

To obtain the particular case stated in the Quest. 1729, let the two arcs VP, VQ be determined by a common focal chord PQ, as shown in the diagram; then $\phi + \phi' = \frac{1}{2}\pi$, and $\tan\phi\tan\phi' = 1$;

$$\therefore\ s_\phi + s_{\phi'} = s_\omega - y_\omega,\quad s_\phi - s_{\phi'} = s_{\omega'} + y_{\omega'},$$

and by (α) and (γ) these neat properties respectively hold good when the amplitudes, or semi-polar angles ω, ω' are determined by the formulæ

$$\tan\omega = \sec\phi + \operatorname{cosec}\phi,\quad \tan\omega' = \sec\phi - \operatorname{cosec}\phi.$$

The former of these, in which we have necessarily $\sec\omega > 3$, is the property enunciated in the question. In the latter, ω' may have any value.

Other curious properties may be deduced from the foregoing general relations. As another example, let $\phi = \phi'$; then

$$\tan\omega'' = \frac{2\sin\phi}{\cos^2\phi}\ \text{and}\ 2s_\phi = s_{\omega''} - y_{\omega''}\tan^2\phi.$$

We may add that, according to (4) the formulæ (α) may be replaced by the corresponding relation

$$\tan(45^\circ + \tfrac{1}{2}\omega) = \tan(45^\circ + \tfrac{1}{2}\phi)\tan(45^\circ + \tfrac{1}{2}\phi')\ \ldots\ldots\ldots\ldots(\alpha),$$

and that, from similar considerations, the formulæ (γ) may be replaced by

$$\tan(45^\circ + \tfrac{1}{2}\omega') = \frac{\tan(45^\circ + \frac{1}{2}\phi)}{\tan(45^\circ + \frac{1}{2}\phi')}\ \ldots\ldots\ldots\ldots(\gamma).$$

These would be more convenient for logarithmic computation.

2114. (Proposed by Rev. J. BLISSARD.)—Prove that $\tan(\cos\theta) =$

$$\frac{\dfrac{\cos\theta}{1} - \dfrac{\cos 3\theta}{1.2.3} + \dfrac{\cos 5\theta}{1.2.3.4.5} - \&c.}{1 - \dfrac{\cos 2\theta}{1.2} + \dfrac{\cos 4\theta}{1.2.3.4} - \&c.} = \frac{\dfrac{\sin 2\theta}{1.2} - \dfrac{\sin 4\theta}{1.2.3.4} + \dfrac{\sin 6\theta}{1.2.3.4.5.6} - \&c.}{\dfrac{\sin\theta}{1} - \dfrac{\sin 3\theta}{1.2.3} + \dfrac{\sin 5\theta}{1.2.3.4.5} - \&c.}$$

Solution by S. W. BROMFIELD; D. M. ANDERSON; *and many others.*

Since $2\cos\theta = \epsilon^{i\theta} + \epsilon^{-i\theta}$ and $2i\sin\theta = \epsilon^{i\theta} - \epsilon^{-i\theta}$, where as usual i stands for $\sqrt{(-1)}$, the *first* fraction becomes, after reduction

$$= \frac{\sin\epsilon^{i\theta} + \sin\epsilon^{-i\theta}}{\cos\epsilon^{i\theta} + \cos\epsilon^{-i\theta}} = \tan\tfrac{1}{2}(\epsilon^{i\theta} + \epsilon^{-i\theta}) = \tan(\cos\theta).$$

The *second* fraction may, in like manner, be also reduced to $\tan(\cos\theta)$.

1787. (Proposed by Professor CREMONA.)—On donne une conique K et un point p. Une transversale menée arbitrairement par p rencontre K en deux points m, m'; et soit x un point de la transversale tel que le rapport anharmonique $(pxmm')$ soit un nombre λ donné. Trouver le lieu du point x. Si λ est l'une des racines cubiques imaginaires de -1, on a une certaine conique $C(p)$. De quelle manière change $C(p)$, si l'on fait varier p?

Recherche analogue par rapport à une surface du second ordre.

Solution by H. R. GREER, B.A.

Take polar coordinates, the point p being the origin, and let the equation of the conic K so referred be $Æ\rho^2 + S\rho + C = 0$; where, for Cartesian coordinates, $Æ = A\cos^2\theta + B\sin^2\theta + 2F\sin\theta\cos\theta$, $S = 2(E\cos\theta + D\sin\theta)$, and C = absolute term. Let the roots of this be ρ_1, ρ_2; form the equation whose root, r, is submitted to the condition $\dfrac{\rho_1(\rho_2 - r)}{\rho_2(r-\rho_1)} = \lambda$, this will be the locus, say L. We can see à priori that it is, generally, of the second degree, and will be unchanged if $\dfrac{1}{\lambda}$ be put for λ. In fact, the equation of L is

$$\left\{(1-\lambda)^2 CÆ + \lambda S^2\right\} r^2 + (1+\lambda)^2 CSr + (1+\lambda)^2 C^2 = 0.$$

If we write $Ær^2 + Sr + C = K$, and $Sr + 2C = P$; so that $K=0$ is the equation of K, and $P = 0$ that of the polar of p with regard to K; it may be thrown into the form $(1-\lambda)^2 CK + \lambda P^2 = 0$, whence it appears that L has double contact with K at the points of contact of tangents drawn from p. Assume $\lambda^2 + 1 = \mu\lambda$, μ being any real constant; then L may be written $(\mu-2)CK + P^2 = 0$; that is to say, L may be real though λ be imaginary, and will be so if the *modulus* of $\lambda\,(=\alpha + i\beta)$ be unity. This happens for the imaginary cube-roots of $+1$. The radius vector will meet L in real points if $(1-\lambda)^2 S^2 - 4(\overline{1-\lambda})^2 CÆ > 0$, that is, meets K and L in real points simultaneously if λ be real, and non-simultaneously if λ be unreal, a result which the equation of the locus *ought* to exhibit.

A precisely similar investigation holds for conicoids. L is determined by λ, and, conversely, λ is determined by L, K and p being given.

1872. (Proposed by Professor CAYLEY.) — Show that the surfaces $xyz = 1$, $yz + zx + xy + x + y + z + 3 = 0$, intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively.

Solution by SAMUEL ROBERTS, M.A.

By elimination of z, we get the conditions

$$\left(1 + x + \frac{1}{y}\right)\left(1 + y + \frac{1}{x}\right) = 0,\ z = \frac{1}{xy};$$

hence the coordinates of points on the curve of intersection must be of the forms

$$x, -\frac{1+x}{x}, -\frac{1}{1+x} \dots (1), \quad x, -\frac{1}{1+x}, \quad -\frac{1+x}{x} \dots (2).$$

Substituting these values in the equation of a plane, we have cubics to determine x. Therefore the curves represented are cubics.

Points belonging to the system or curve (1) lie on the cubic cone

$$z^2y + y^2x + x^2z - 3xyz = 0;$$

those belonging to the system (2) lie on the cubic cone

$$z^2x + x^2y + y^2z - 3xyz = 0.$$

It is easy to see that if (a, b, c) satisfy the given equations, they are also satisfied by the six permutations of (a, b, c) and the six permutations of $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. For a given value of x, three of the direct permutations belong to (1), and three belong to (2). In the case of the inverse permutations the order is reversed.

1892. (Proposed by Professor SYLVESTER.)—Observing that the form

$$\left\{(b^2-a^2)x + bcy + c^2z\right\}^2 - 2d^2\left\{(b^2+a^2)x + bcy + c^2z\right\}z + d^4z^2$$

is a function solely of $x, y, z, a^2-b^2, c^2-d^2, bc, ad$; show that if

$$\alpha^2 = \frac{b^2-a^2}{c^2-d^2}d^2, \quad \beta^2 = \frac{b^2-a^2}{c^2-d^2}c^2, \quad \gamma^2 = \frac{c^2-d^2}{b^2-a^2}b^2, \quad \delta^2 = \frac{c^2-d^2}{b^2-a^2}a^2,$$

and A, B, B′ be three points in a straight line such that $\text{AB} = \frac{c}{b}$, $\text{AB}' = \frac{\gamma}{\beta}$; then, if any point P be found satisfying the equation $a \,.\, \text{AP} + b \,.\, \text{BP} = d$, on giving right signs to α, β, the equation $\alpha \,.\, \text{AP} + \beta \,.\, \text{B}'\text{P} = \delta$ will also be satisfied.

Solution by W. H. LAVERTY; S. W. BROMFIELD; *and others.*

We see that $a^2-b^2 = \alpha^2-\beta^2$, $c^2-d^2 = \gamma^2-\delta^2$, $bc = \beta\gamma$, $ad = \alpha\delta$;

$\therefore$ $\text{F}(x, y, z, a^2-b^2, c^2-d^2, bc, ad) = \text{F}(x, y, z, \alpha^2-\beta^2, \gamma^2-\delta^2, \beta\gamma, \alpha\delta)$.

Let $$\text{AP}^2 = \frac{x}{z} = \frac{(-y)^2}{4z^2},$$

then $$\text{BP}^2 = (\text{AP}-\text{AB})^2 = \left(\frac{-y}{2z} - \frac{c}{b}\right)^2 = \frac{x}{z} + \frac{c^2}{b^2} + \frac{cy}{bz}.$$

Now $$a^2 \,.\, \text{AP}^2 + b^2 \,.\, \text{BP}^2 + 2ab \,.\, \text{AP} \,.\, \text{BP} - d^2 = 0;$$

$$\therefore \quad (a^2+b^2)\frac{x}{z} + b^2\left(\frac{c^2}{b^2} + \frac{cy}{bz}\right) + 2ab\left(\frac{-y}{2z} - \frac{c}{b}\right)\left(\frac{-y}{2z}\right) - d^2 = 0;$$

therefore $$(b^2+a^2)x + c^2z + bcy - d^2z = -a(2bx+cy);$$

therefore $$\left\{(b^2+a^2)x + c^2z + bcy - d^2z\right\}^2 = 4a^2(b^2x^2 + bcxy + c^2xz),$$

or $\quad \left\{(b^2-a^2)\,x+bcy+c^2z\right\}^2-2d^2\left\{(b^2+a^2)\,x+bcy+c^2z\right\}z+d^4z^2=0;$

$\therefore$ also $\left\{(\beta^2-a^2)\,x+\beta\gamma y+\gamma^2z\right\}^2-2\delta^2\left\{(\beta^2+\alpha^2)x+\beta\gamma y+\gamma^2z\right\}z+\delta^4z^2=0;$

therefore $\quad (\beta^2+\alpha^2)\,x+\beta\gamma y+\gamma^2z-\delta^2z=-\alpha\,(2\beta x+\gamma y);$

therefore $\alpha^2\,.\,\frac{x}{z}+\beta^2\left\{\frac{x}{z}+\frac{\gamma}{\beta}\,.\,\frac{y}{z}+\frac{\gamma^2}{\beta^2}\right\}+2\alpha\beta\left(\frac{x}{z}\right)^{\frac{1}{2}}\left\{\left(\frac{x}{z}\right)^{\frac{1}{2}}+\frac{\gamma}{\beta}\right\}=\delta^2;$

therefore $\quad \alpha^2\,.\,\text{AP}^2+\beta^2\,(-\text{AP}+\text{B}'\text{A})^2+2\alpha\beta\,.\,\text{AP}\,.\,(\text{AP}-\text{B}'\text{A})=\delta^2;$

therefore $\quad \alpha\,.\,\text{AP}+\beta\,.\,\text{BP}'=\delta.$

1969. (Proposed by Professor SYLVESTER.)—In two given great circles of a sphere intersecting at O are taken respectively two points P and Q, the arc joining which is of given length: prove that S, H two fixed points, and M a fixed line, in a plane may be found such that, for all positions of the arc PQ, a point M in the fixed line may be found satisfying the equations

$$\text{SM} \pm \text{HM} = \sin \text{OP}, \quad \text{SM} \mp \text{HM} = \sin \text{OQ}.$$

Solution by PROFESSOR CAYLEY.

1. In the spherical triangle OPQ, whereof the sides OP, OQ, PQ are θ, ϕ, β and the angle O is $=\alpha$, the relation between these quantities is $\cos\alpha=\dfrac{\cos\beta-\cos\theta\cos\phi}{\sin\theta\sin\phi}$; hence treating α, β as constants, and θ, ϕ as variable angles connected by the foregoing equation, it is required to show that we can find two fixed points S, H and a fixed line, such that taking M a variable point in this line and writing $\text{SM}=r$, $\text{HM}=s$, the relation between r and s (or equation of the fixed line in terms of r, s as coordinates of a point thereof) is obtained by substituting in the foregoing equation for θ and ϕ the values given by the two equations

$$\sin\theta=(r+s), \quad \sin\phi=(r-s),$$

or, as for the sake of homogeneity, it will be more convenient to write these equations, $\quad m\sin\theta=(r+s), \quad m\sin\phi=(r-s).$

2. Suppose that the perpendicular distances of S, H from the fixed line are a and b, and that the distance between the feet of the two perpendiculars is $2c$, then if x denote the distance of the point M from the midway point between the feet of the two perpendiculars, we have

$$r=\surd\left\{(c+x)^2+a^2\right\}, \quad s=\surd\left\{(c-x)^2+b^2\right\},$$

and (a, b, c) being properly determined, the elimination of x from these equations should give between (r, s) a relation equivalent to that obtained by the elimination of (θ, ϕ) from the before mentioned equations. Or, what is the same thing, the elimination of (r, s, x) from the equations

$m \sin\theta = r+s, \quad m\sin\phi = r-s, \quad r=\surd\{(c+x)^2+a^2\}, \quad s=\surd\{(c-x)^2+b^2\}$
should give between (θ, ϕ) the relation
$\cos\alpha = \dfrac{\cos\beta - \cos\theta\cos\phi}{\sin\theta\sin\phi}$; that is, the last mentioned equation should be obtained by the elimination of x from the equations
$m(\sin\theta+\sin\phi)=2\surd\{(c+x)^2+a^2\}, \; m(\sin\theta-\sin\phi)=2\surd\{(c-x)^2+b^2\}.$

3. The equation in (θ, ϕ) may be written

$$\cos\beta - \cos\alpha\sin\theta\sin\phi = \cos\theta\cos\phi,$$

or squaring and reducing

$$\sin^2\theta+\sin^2\phi = \sin^2\beta + 2\cos\alpha\cos\beta\sin\theta\sin\phi + \sin^2\alpha\sin^2\theta\sin^2\phi,$$

that is,

$$\sin^2\theta+\sin^2\phi = \frac{1-\cos^2\alpha-\cos^2\beta}{\sin^2\alpha} + \left(\sin\alpha\sin\theta\sin\phi + \frac{\cos\alpha\cos\beta}{\sin\alpha}\right)^2.$$

But from the two equations in x, we have

$$m^2(\sin^2\theta+\sin^2\phi) = 4c^2+2a^2+2b^2+4x^2, \quad m^2\sin\theta\sin\phi = 4cx+a^2-b^2,$$

whence $$2x = \frac{b^2-a^2+m^2\sin\theta\sin\phi}{2c},$$

therefore $$\sin^2\theta+\sin^2\phi = \frac{4c^2+2b^2+2a^2}{m^2} + \left(\frac{b^2-a^2+m^2\sin\theta\sin\phi}{2cm}\right)^2.$$

Hence, comparing the two results, we have

$$\frac{1-\cos^2\alpha-\cos^2\beta}{\sin^2\alpha} = \frac{4c^2+2b^2+2a^2}{m^2}, \quad \frac{\cos\alpha\cos\beta}{\sin\alpha} = \frac{b^2-a^2}{2cm}, \quad \sin\alpha = \frac{m}{2c};$$

or, as these may also be written,

$$\sin\alpha = \frac{m}{2c}, \quad \cos^2\alpha+\cos^2\beta = \frac{-b^2-a^2}{2c^2}, \quad 2\cos\alpha\cos\beta = \frac{b^2-a^2}{2c^2};$$

whence $$(\cos\alpha+\cos\beta)^2 = \frac{-a^2}{c^2}, \quad (\cos\alpha-\cos\beta)^2 = \frac{-b^2}{c^2}, \quad \sin\alpha = \frac{m}{2c};$$

so that m being put equal to unity, or otherwise assumed at pleasure, a, b, c are given functions of α, β. Or conversely, if a, b, c are assumed at pleasure, then α, β, m are given functions of these quantities.

5. It is to be remarked that (α, β) being real, a and b will be imaginary, and consequently the points S, H of Professor SYLVESTER's theorem are imaginary;* if, however, we write $-a^2, -b^2$ in place of a^2, b^2 respectively, then the radicals $\surd\{(c+x)^2-a^2\}, \surd\{(c-x)^2-b^2\}$ have a real geometrical interpretation. The system of relations between $(\alpha, \beta, a, b, c, m)$ becomes

$$(\cos\alpha+\cos\beta)^2 = \frac{a^2}{c^2}, \quad (\cos\alpha-\cos\beta)^2 = \frac{b^2}{c^2}, \quad \sin\alpha = \frac{m}{2c};$$

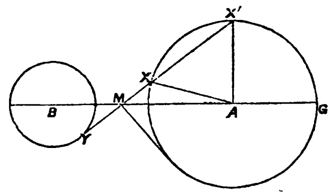
and considering (a, b, c) as given, we may write

$$\cos\alpha = \frac{a+b}{2c}, \quad \cos\beta = \frac{a-b}{2c}, \quad m = \surd\{4c^2-(a+b)^2\},$$

[* Prof. SYLVESTER remarks that according as β is less or greater than α, we may find real values of θ, ϕ equal to one another in the one case and supplementary in the other. Hence we must in any case be able to make $r=0$ and $s=0$ indifferently, which shows *à priori* that the line being supposed real, each point S, H must be imaginary, but so that the squared distance of either from the line must be a *real negative quantity*, conformably to Prof. CAYLEY's important observation in the text.]

viz., we have either this system or the similar system obtained by writing $-b$ in place of b.

6. Consider two circles with the radii a, b and having the distance of their centres $= 2c$, and to fix the ideas assume that $2c > a+b$, that is, that the circles are exterior to each other. The foregoing equations signify that $90^\circ - \alpha$, $90^\circ - \beta$ are the inclinations to the line of centres of the inverse and the direct common tangents respectively, and that m is the length of the inverse common tangent. And the theorem is, that considering two circles as above, and taking M a variable point in the line of centres, if r, s denote the tangential distances of m from the two circles respectively, and if m be the length of the inverse common tangent of the two circles, then the angles θ, ϕ determined by the equations



$$m \sin\theta = r+s, \quad m\sin\phi = r-s,$$

are connected by the relation

$$\cos\beta = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\alpha,$$

(α, β) being constant angles, determined as above.

7. It is to be remarked that, assuming $k = \dfrac{\sin\alpha}{\sin\beta} = \dfrac{\sqrt{\left\{4c^2-(a+b)^2\right\}}}{\sqrt{\left\{4c^2-(a-b)^2\right\}}}$,

that is, $k =$ inverse common tangent $\div$ direct common tangent, then we have $\cos\alpha = \sqrt{(1-k^2\sin^2\beta)} = \Delta\beta$, or the equation in θ, ϕ becomes

$$\cos\beta = \cos\theta\cos\phi + \sin\theta\sin\phi\Delta\beta,$$

which is the algebraical equation connecting the amplitudes of the elliptic functions in the relation $\mathrm{F}(\theta) + \mathrm{F}(\phi) = \mathrm{F}(\beta)$.

8. It is very noticeable that the above figure leads to another relation in elliptic functions, viz., it is the very figure employed for that purpose by Jacobi; in fact, considering therein YM as a variable tangent meeting the circle A in the two points X and X′, then if 2ψ, $2\psi'$ denote the angles GAX, GAX′ respectively, it is easy to see geometrically that we have $d\psi : d\psi' = \mathrm{YX} : \mathrm{YX}'$; where $(\mathrm{YX})^2 = (\mathrm{BX})^2 - b^2 = 4c^2 + a^2 + 4ac\cos 2\psi - b^2 = (2c+a)^2 - b^2 - 8ac\sin^2\psi$, and similarly $(\mathrm{YX}')^2 = (2c+a)^2 - b^2 - 8ac\sin^2\psi'$, that is, writing $l^2 = \dfrac{8ac}{(2c+a)^2-b^2}$, the differential equation is

$$\frac{d\psi}{\sqrt{(1-l^2\sin^2\psi)}} - \frac{d\psi'}{\sqrt{(1-l^2\sin^2\psi')}} = 0,$$

corresponding to an integral equation $\mathrm{F}(\psi) - \mathrm{F}(\psi') = \mathrm{F}(\mu)$,

the modulus of the elliptic functions being $\quad l, = \dfrac{\sqrt{8ac}}{\sqrt{\left\{(2c+a)^2 - b^2\right\}}}$.

In the problem above considered the modulus is $\quad k, = \dfrac{\sqrt{\left\{4c^2-(a+b)^2\right\}}}{\sqrt{\left\{4c^2-(a-b)^2\right\}}}$,

and it is not very easy to see the connexion between the two results.

1940. (Proposed by the Rev. J. BLISSARD.)—
Given that $1.2.3\ldots x\ (x \text{ inf.})\ \sqrt{(2\pi)}\ x^{x+\frac{1}{2}}\epsilon^{-x}$, prove that

$$m(m+n)\ldots\left\{m+(x-1)n\right\}\ (x \text{ inf.}) = \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{m}{n}\right)}\, n^x x^{x+\frac{m}{n}-\frac{1}{2}}\epsilon^{-x}.$$

Solution by the PROPOSER.

In $1.2.3\ldots.x\ (x \text{ inf.}) = \sqrt{(2\pi)}\, x^{x+\frac{1}{2}}\epsilon^{-x}$ put $x+m-1$ for x, and divide by $1.2\ldots.(m-1)$, which $=\Gamma(m)$; then we have

$$m(m+1)\ldots.(m+x-1) = \frac{\sqrt{(2\pi)}}{\Gamma(m)}(x+m-1)^{x+m-\frac{1}{2}}\epsilon^{-(x+m-1)}.$$

Taking logarithms of each side, we have

$$\log\left\{m(m+1)\ldots.(m+x-1)\right\}$$
$$= \log\left\{\frac{\sqrt{(2\pi)}}{\Gamma(m)}\right\}+(x+m-\tfrac{1}{2})\left(\log x+\frac{m-1}{x}+\&c.\right)-(x+m-1)$$
$$= \log\left\{\frac{\sqrt{(2\pi)}}{\Gamma(m)}\right\}+(x+m-\tfrac{1}{2})\log x - x;$$

therefore $\qquad m(m+1)\ldots.(m+x-1) = \dfrac{\sqrt{(2\pi)}}{\Gamma(m)}\, x^{x+m-\frac{1}{2}}\epsilon^{-x};$

and putting $\frac{m}{n}$ for m, and multiplying by n^x, we have

$$m(m+n)\ldots.\left\{m+(x-1)n\right\} = \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{m}{n}\right)}\, n^x x^{x+\frac{m}{n}-\frac{1}{2}}\epsilon^{-x}.$$

1990. (Proposed by Professor SYLVESTER.)—Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

Solution by M. W. CROFTON, B.A.

This singular theorem, of which Professor CAYLEY has given an instructive discussion by rectangular coordinates on pp. 35—39 of Vol. VI. of the *Reprint*, admits of a simple proof depending entirely upon the focal properties of the curves.

I. Given three points on a Cartesian oval 1, 2, 3, and one focus F, to find the locus of the two other foci.

Let ρ_1, ρ_2, ρ_3 be the distances of 1, 2, 3 from F, and $\sigma_1, \sigma_2, \sigma_3$ their distances from G, another focus: we shall have

$$\rho_1+k\sigma_1 = l, \quad \rho_2+k\sigma_2 = l, \quad \rho_3+k\sigma_3 = l;$$

eliminate k, l, and we find

$$(\rho_2-\rho_3)\,\sigma_1+(\rho_3-\rho_1)\,\sigma_2+(\rho_1-\rho_2)\,\sigma_3=0 \text{ ....(1)},$$

showing clearly that the locus required is a circular cubic passing through F, and having 1, 2, 3 as concyclic foci.

Also, *if* 1, 2, 3 *are any three points on a Cartesian, its foci lie on some circular cubic, of which* 1, 2, 3 *are concyclic foci.*

II. Suppose now that 1, 2, 3, 4 are the four concyclic foci of a circular cubic; let A, B be any two points on the curve, draw a Cartesian having A, B as two foci, passing through 1 and 2 (there is but one such Cartesian); let $\rho_1, \rho_2, \rho_3, \rho_4$ be the vectors from A to 1, 2, 3, 4; $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ from B. Suppose the equation of the Cartesian to be

$$a\rho+b\sigma=1 \text{ ....(2)}.$$

Since (ρ_1, σ_1) and (ρ_2, σ_2) both satisfy this equation, the point (ρ_3, σ_3) will also satisfy it; for by the (focal) definition of the circular cubic

$$\rho_3=\frac{m\rho_1+n\rho_2}{m+n},\quad \sigma_3=\frac{m\sigma_1+n\sigma_2}{m+n};$$

and these values evidently satisfy (2).

Hence the Cartesian passes through the focus 3 of the cubic, and also through 4.

Therefore, taking any two points A, B on the circular cubic, a Cartesian can be drawn with A, B as two foci, and passing through the 4 concyclic foci of the cubic. The third focus of the Cartesian will be the point in which AB produced meets the cubic again, as is evident from (I). This proves Professor SYLVESTER'S theorem.

The following remarkable property follows at once from the above:—*If three points* A, B, C *on a Cartesian be given, and one focus F, the family of Cartesians all pass through a 4th fixed point.* This point will be the 4th focus of the circular cubic through F with A, B, C as foci. For, from (I.), the three foci of the Cartesian lie on this cubic, and therefore from (II.) it is clear that it passes through the 4th focus of the cubic, as well as through A, B, C.

1994. (Proposed by M. W. CROFTON, B.A.)—Two circles have double internal contact with an ellipse, and a third circle passes through the four points of contact. If t, t', T be the tangents from any point on the ellipse to these three circles, prove that $T^2=tt'$.

I. *Solution by the* REV. R. TOWNSEND, F.R.S.

If the equations of the three circles be respectively $s=0$, $s'=0$, and $S=0$, then that of the ellipse, as touching s and s' at their intersections with S, being necessarily $ss'=S^2$, therefore, &c.

N.B.—That the equation $ss'=S^2$ should reduce to the second order, when s, s', and S are any three circles, the centre of S must be collinear with and lie midway between those of s and s'; hence, if a conic have double contact with two circles, the centre of the circle determined by the four points of contact lies midway between those of the touched circles.

II. *Solution by* W. H. Laverty; H. Tomlinson; J. Dale; W. Chadwick; *and others.*

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, be the equation to the ellipse; $(d, 0)$ and $(x' y')$ the coordinates respectively of the centre, and one point of contact of one of the circles; r its radius: then its equation is $(x-d)^2 + y^2 = r^2$, and therefore the equation of the common tangent must be of both the forms

$$(x-d)(x'-d) + yy' = r^2, \text{ and } \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1;$$

$$\therefore \frac{a^2}{x'}(x'-d) = b^2 = r^2 + d(x'-d); \ \therefore x' = \frac{d}{e^2}, \text{ and } r^2 = b^2 - \frac{b^2 d^2}{a^2 - b^2}.$$

The length of the tangent to this circle from the point $\left(x, \frac{b}{a}\sqrt{(a^2-x^2)}\right)$ is given by

$$t^2 = (x-d)^2 + \frac{b^2}{a^2}(a^2-x^2) - b^2 + \frac{b^2 d^2}{a^2-b^2} = \left(ex - \frac{d}{e}\right)^2.$$

Similarly we find that $\quad t' = ex - \frac{d'}{e}$.

$$\text{Now } T^2 = \left(x - \frac{d+d'}{2}\right)^2 + \frac{b^2}{a^2}(a^2-x^2) - \left\{\left(\frac{d+d'}{2} - \frac{d}{e^2}\right)^2 + b^2 - \frac{b^2}{a^2}\left(\frac{d}{e^2}\right)^2\right\}$$

$$= e^2x^2 - (d+d')x + \frac{dd'}{e^2} = \left(ex - \frac{d}{e}\right)\left(ex - \frac{d'}{e}\right)$$

$= t \, . \, t'$; which proves the theorem.

On some Problems in the Theory of Chances.
By Hugh Godfray, M.A.

The following remarks may serve to show why discordant results are sometimes obtained in treating questions of probability, such as the four-point problem (*Reprint*, Vol. V., p. 81).

I believe it will be found that the discordance arises from the fact, that the word *random* is not sufficiently defined in the question; and the possibility of considering it in different ways, makes so many different problems of which the various results are solutions.

Perhaps my meaning will be made clearer by an example, and the following will answer the purpose:—Two chords are drawn at random in a circle, what is the chance that they will intersect?

Now, what is a chord drawn at random?

(1.) We may consider the circumference of the circle to be divided into a great number of very small equal arcs, and that by a chord drawn at random we mean a chord joining any two of these arcs,—*all combinations being equally probable.*

(2.) We may consider that a random chord means a line whose distance

from the centre is less than the radius,—*all such distances being equally probable.*

(3.) A random chord may mean any line joining two points of the circle, *all lengths less than the diameter being equally probable.*

&c. &c.

It will be easy to show that the average length of such chords in Case (2) will be greater than in (1), but that in (3) they will be much less.

To continue the discussion of the problem :—Two such chords are said to be drawn at random. Here we have another element of uncertainty introduced, which can only be removed by some further limitation. The limitation which is usually, but tacitly, supposed, is that the second chord may be inclined at any angle to the first, and that *all inclinations are equally probable* ; but it would not be difficult to give other meanings to the random relative positions of the two chords.

Now assuming all inclinations of the two chords to be equally probable, it will be found that the chance of the two chords intersecting will be $\frac{1}{3}$, when the interpretation of a random chord is that of Case (1); it will be $\frac{1}{2}$ in Case (2); and $\dfrac{3\pi-8}{4\pi}$ in Case (3).

Case (1) is the view Mr. WOOLHOUSE has taken in his solution of the more general Question 1894. (*Reprint,* Vol. V., p. 110.)

Case (2) was set by Professor ADAMS in the *Smith's Prize Examination,* in 1865.

Case (3) I have not met with before, and as a simple exercise in integration it is proposed for solution as Question 2263.

I shall now consider the solutions of the four-point problem on p. 81, Vol. V., and it will be seen that Mr. WILSON and Dr. INGLEBY have solved two different problems.

Taking a point at random, according to Dr. INGLEBY, is supposing the unlimited area to be divided into an infinite number of indefinitely small *equal* areas, and assuming that the point is equally likely to occupy any one of these; so that the chance of its falling within any given area δ is $\dfrac{\delta}{\text{whole area}}$. This is a satisfactory definition, and if legitimately followed out will give $\frac{1}{2}$ for the probability, because $\epsilon+\phi-\delta$ is indefinitely small compared with $\alpha+\beta+\gamma$.

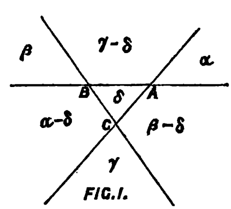


FIG. 1.

If ABC (Fig. 1) be the triangle formed by joining three of the points; and, δ being the area of the triangle, if we call α the infinite area contained between the lines AB and AC indefinitely produced &c., the probability will be

$$\frac{\alpha+\beta+\gamma+\delta}{2(\alpha+\beta+\gamma)-2\delta}=\frac{1}{2}+\frac{\delta}{\alpha+\beta+\gamma-\delta}=\frac{1}{2},$$

since δ is indefinitely small compared with $\alpha+\beta+\gamma$.

Mr. WILSON'S definition of a random point is, that it is the intersection of two random lines; and from this definition he has correctly obtained the value $\frac{1}{3}$.

The special law according to which these random lines are drawn will not affect the result.

If we suppose two points A and B (Fig. 2) to have been determined, and that a third point C is required by the intersection of two random lines through A and B. We may suppose round each of the points an indefinite number of these random lines, and the small quadrilateral spaces formed may be considered as each equally likely to be occupied by a random point. But these areas are clearly not equal, and therefore a random point according to this definition is not the same as according to the former, and the probability may therefore be expected to be different.

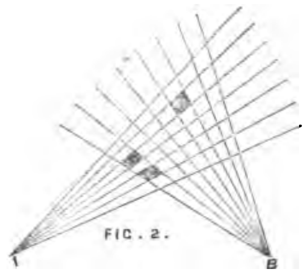


FIG. 2.

I have no doubt that the various other solutions referred to by Dr. INGLEBY may be reconciled in the same say.

1996. (Proposed by W. K. CLIFFORD.)—If four circles A = 0, B = 0, C = 0, D = 0 are mutually orthotomic, the square of the radius of a circle $lA+mB+nC+sD=0$ is $(l^2r_1^2+m^2r_2^2+n^2r_3^2+s^2r_4^2)\div(l+m+n+s)^2$, where r_1, r_2, r_3, r_4 are the radii of A, B, C, D.

I. *Solution by* W. H. LAVERTY; H. TOMLINSON; W. CHADWICK; *and others.*

If (α_1, β_1), (α_2, β_2) be the centres of two of the circles, we must have

$$(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2=r_2^2+r_1^2 \quad\ldots\ldots\ldots\ldots (a),$$

and five other similar conditions.

Now dividing $lA+mB+nC+sD=0$ by $(l+m+n+s)$, and writing it in the form $(x-\alpha)^2+(y-\beta)^2=r^2$, we find

$$r^2=\frac{\{\Sigma(l\alpha_1)\}^2+\{\Sigma(l\beta_1)\}^2}{\{\Sigma(l)\}^2}+\frac{\Sigma(lr_1^2)-\Sigma\left(l(\alpha_1^2+\beta_1^2)\right)}{\Sigma(l)}$$

$$=\frac{\Sigma(l^2r_1^2)+\Sigma\left(lm\left[2(\alpha_1\alpha_2+\beta_1\beta_2)+r_2^2+r_1^2-(\alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2)\right]\right)}{\{\Sigma(l)\}^2}.$$

But by the condition (a) the second term of the numerator vanishes,

therefore $$r^2=\frac{l^2r_1^2+m^2r_2^2+n^2r_3^2+s^2r_4^2}{(l+m+n+s)^2}.$$

II. *Solution by* J. DALE; H. MURPHY; *and others.*

Let the equations of the circles be

(A) .... $(x-a_1)^2+(y-b_1)^2-r_1^2=0$ | (C).... $(x-a_3)^2+(y-b_3)^2-r_3^2=0$
(B) .... $(x-a_2)^2+(y-b_2)^2-r_2^2=0$ | (D).... $(x-a_4)^2+(y-b_4)^2-r_4^2=0$

If these circles are mutually orthotomic, we must have the following conditions :—

$$(a_1-a_2)^2+(b_1-b_2)^2 = r_1^2+r_2^2 \ldots. (1) \quad (a_2-a_3)^2+(b_2-b_3)^2 = r_2^2+r_3^2 \ldots. (4)$$
$$(a_1-a_3)^2+(b_1-b_3)^2 = r_1^2+r_3^2 \ldots. (2) \quad (a_2-a_4)^2+(b_2-b_4)^2 = r_2^2+r_4^2 \ldots. (5)$$
$$(a_1-a_4)^2+(b_1-b_4)^2 = r_1^2+r_4^2 \ldots. (3) \quad (a_3-a_4)^2+(b_3-b_4)^2 = r_4^2+r_4^2 \ldots. (6)$$

The equation of the circle $lA+mB+nC+sD=0$ is

$$(l+m+n+s)(x^2+y^2)-2(la_1+ma_2+na_3+sa_4)x-2(lb_1+mb_2+nb_3+sb_4)y$$
$$+[l(a_1^2+b_1-r_1^2)+m \text{ \&c.}] = 0,$$

and the square of the radius of this circle is equal to

$$\frac{(la_1+ma_2+na_3+sa_4)^2+(lb_1+mb_2+nb_3+sb_4)^2}{(l+m+n+s)^2}$$
$$-\frac{(l+m+n+s)\left\{l(a_1^2+b_1^2-r_1^2)\right\}+\text{\&c.}}{(l+m+n+s)^2}$$

where the coefficients of l^2, m^2, n^2, s^2 are respectively r_1^2, r_2^2, r_3^2, r_4^2; also the coefficient of any of the other terms, such as lm, is

$$r_1^2+r_2^2+2(a_1a_2+b_1b_2)-(a_1^2+b_1^2+a_2^2+b_2^2)$$
$$= r_1^2+r_2^2-\left\{l(a_1-a_2)^2+(b_1-b_2)^2\right\}=0, \text{ by (1).}$$

Similarly the coefficients of ln, ls, &c., vanish; hence the expression for the square of the radius reduces to the form given in the question.

2001. (Proposed by W. GODWARD.)—If r and r_1 be the radii of two circles each having double contact with a conic, the former passing through the centre of the conic, and the latter through one of the foci; prove that

$$r : r_1 = a : 2b.$$

Solution by J. DALE; W. CHADWICK; H. TOMLINSON; W. H. LAVERTY *and others.*

Taking the case of the ellipse; the circles of double contact, passing, the one through the centre, and the other through a focus, must touch the ellipse internally, and have their centres on the major axis. Taking the origin at the centre, let (x, y) be the point of contact of the circle (r) with the conic; then we have

$$r^2 = (\text{normal})^2 = y^2+(1-e^2)^2x^2 = (1-e^2)(a^2-e^2x^2) = e^4x^2,$$

therefore $\quad a^2-e^2x^2 = a^2e^2$, and $r^2 = a^2e^2(1-e^2) = e^2b^2$.

Again, let (x, y) be the point of contact of the circle (r_1) with the conic; then

$$r_1^2 = (\text{normal})^2 = (1-e^2)(a^2-e^2x^2) = e^2(a-ex)^2,$$

therefore $(a^2-e^2x^2) = 4a^2e^2(1-e^2)$ and $r_1^2 = 4a^2e^2(1-e^2)^2 = 4e^2\dfrac{b^4}{a^2}$,

therefore $\quad r : r_1 = a : 2b.$

In the case of the hyperbola, the value for r^2 shows that the construction is impossible. The circle in this case must touch the hyperbola externally and have its centre on the minor axis, so that the above equation does not remain true. If, however, we take the circle r as the one having double contact with the *conjugate* hyperbola, and passing through the centre, we shall still have the relation $r : r_1 = a : 2b$.

II. *Solution by the* PROPOSER.

It is evident from the symmetry of the curves that the centres of the two circles will be in the direction of the transverse axis of the conic; taking therefore the centre of the conic for the origin, and its principal diameters as axes of coordinates, the equation of the circle whose radius is r is

$$y^2 - 2rx + x^2 = 0 \ldots\ldots (1);$$

and the coordinates of the centre of the circle whose radius is r_1 are $(ae \mp r_1, 0)$, so that its equation is

$$x^2 + y^2 - 2(ae \mp r_1)x + a^2e^2 \mp 2aer_1 = 0 \ldots. (2),$$

the upper signs being taken when the conic is an ellipse, and the lower when an hyperbola.

Also the equation to the ellipse and *conjugate* hyperbola is

$$\frac{b^2x^2}{a^2} \pm y^2 \mp b^2 = 0 \ldots\ldots (3),$$

and the equation to the ellipse and hyperbola

$$\frac{b^2x^2}{a^2} \pm y^2 - b^2 = 0 \ldots\ldots (4).$$

Eliminating y^2 between (1) and (3), we have

$$e^2x^2 - 2rx + b^2 = 0;$$

hence in order that (1) and (3) should have contact we must have $r^2 = b^2e^2$, or $r = be$.

Likewise eliminating y^2 between (2) and (4), we have

$$e^2x^2 - 2(ae \mp r_1)x + a^2e^2 \mp 2aer_1 \pm b^2 = 0,$$

$$\text{or,} \quad e^2x^2 - 2(ae \mp r_1)x + a^2 \mp 2aer_1 = 0 \ldots\ldots (5),$$

so that for contact of (2) and (4), we must have

$$(ae \mp r_1)^2 = e^2(a^2 \mp 2aer_1), \text{ which gives } r_1 = \pm 2ae(1-e^2) = \frac{2b^2e}{a};$$

therefore $$r : r_1 = a : 2b.$$

COR.—Let P (x, y) be the point of contact of (2) and (4); then if we substitute $r_1 = \pm 2ae(1-e^2)$ in (5), we shall finally obtain $x = \frac{a}{e}(2e^2-1)$; also from (4),

$$y^2 = \pm b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{a^2(1-e^2)}{e^2}(-1 + 5e^2 - 4e^4) = \frac{a^2}{e^2}(-1 + 6e^2 - 9e^4 + 4e^6).$$

Let F be the focus through which circle radius r_1 passes, then its coordinates are $(ae, 0)$. These coordinates of F and P, give FP $=$ $\pm 2a(1-e^2) =$ latus rectum of the conic. This corollary furnishes a Solution to a question in the Senate House Examinations for 1865, where it was proposed for the ellipse. The property also holds for the parabola.

1865. (Proposed by P. O'CAVANAGH.)—Find the equation and parameter of the parabola osculating most closely at the origin the conic

$$ax + by + cx^2 + 2dxy + ey^2 = 0 \ldots\ldots (1);$$

and find also the angle (θ) between the axis of x and the axis of the required parabola.

Solution by the REV. J. L. KITCHIN, M.A.; S. W. BROMFIELD; W. H. LAVERTY; *and others.*

1. The equation of the required parabola may be written

$$ax + by + h^2x^2 + 2hkxy + k^2y^2 = 0 \ldots\ldots (2),$$

$ax + by = 0$ being the common tangent at the origin to (1) and (2), and $hx + ky = 0$ the diameter of (2) through the origin.

Subtracting (1) from (2) we get

$$(h^2 - c)\, x^2 + 2\,(hk - d)\, xy + (k^2 - e)\, y^2 = 0 \ldots\ldots (3),$$

which must represent a pair of common chords of (1) and (2) passing through the origin. Now in order that the parabola (2) may have the closest possible (4-pointic) contact with the given conic (1), the equation (3) must be identical with $(ax + by)^2 = 0$, hence we must have

$$h^2 - c = la^2,\ \ hk - d = lab,\ \ k^2 - e = lb^2;\ \ \therefore (la^2 + c)(lb^2 + e) = (lab + d)^2,$$

whence $\quad l = \dfrac{ce - d^2}{\text{D}}$, where $\text{D} = 2abd - a^2e - b^2c$; and then we have

$$h^2 = \frac{(ad - bc)^2}{\text{D}},\ \text{and}\ k^2 = \frac{(ae - bd)^2}{\text{D}}.$$

Hence the equation to the osculating parabola at the origin is

$$\left\{(ad - bc)\, x + (ae - bd)\, y\right\}^2 = \text{D}\,(ax + by).$$

2. *Otherwise:* the equation to the tangent at the origin being $ax + by = 0$, the equation to the parabola must be

$$k\,(ax + by + cx^2 + 2dxy + ey^2) - (ax + by)^2 = 0;$$

and that this may be a parabola, we must have

$$(kd - ab)^2 = (kc - a^2)(ke - b^2),\ \text{whence}\ k = \frac{2dab - (a^2e + b^2c)}{d^2 - ce};$$

and the equation to the parabola is as given in Art. 1.

3. Now $(ad - bc)\, x + (ae - bd)\, y = 0$ is the equation to the diameter of the parabola through the origin, and the diameter is parallel to the axis; therefore we have $\quad \tan\theta = \dfrac{ad - bc}{bd - ae}$.

4. We easily find the equation to the directrix; it is

$$(ae - bd)\, x - (ad - bc)\, y = \tfrac{1}{4}\,(a^2 + b^2).$$

The perpendicular from the origin on this line is equal to the focal distance of the origin, and the parameter (p') of the diameter through the origin is four times this distance,

therefore $\quad p' = \dfrac{a^2 + b^2}{\sqrt{\left\{(ad - bc)^2 + (ae - bd)^2\right\}}}.$

The tangent at the origin to the parabola is $ax + by = 0$.

If ϕ be the angle between this line and the diameter of the parabola, then the latus rectum or principal parameter $p = p' \sin^2 \phi$.

We find, $\sin^2 \phi = \dfrac{D^2}{(a^2+b^2)\left\{(ad-bc)^2+(ae-bd)^2\right\}}$,

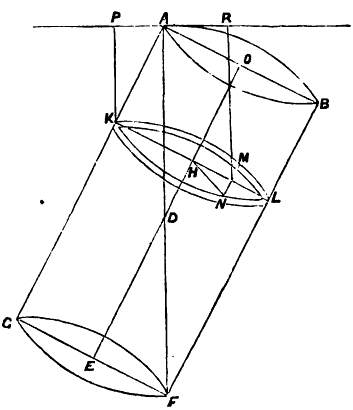
therefore $$p = \frac{D^2}{\left\{(ad-bc)^2+(ae-bd)^2\right\}^{\frac{3}{2}}}.$$

It is now easy to find the coordinates of the focus and vertex of the parabola; and consequently the equation to the axis of the parabola.

[From the above value for $\tan\theta$ it is clear that the axis of the parabola is parallel to the diameter of the given conic through the point of contact. The same property is proved in the *Lady's and Gentleman's Diary* for 1859 (pp. 65—70), where it was proposed to find "the locus of the focus of a varying parabola osculating most closely a given ellipse or hyperbola, at the various points of the given conic."]

1674. (Proposed by J. W. T. BLAKEMORE, B.A.)—A cylinder filled with fluid is closed at both ends, and then suspended from a point in the rim of one end; find the resultant pressure on the curved surface, and prove that the direction of this resultant pressure, and the axis of the cylinder, make equal angles with the vertical and horizontal respectively.

Solution by the REV. J. L. KITCHIN, M.A.



Let A be the point of suspension of the cylinder. It is evident that the centre of gravity of the cylinder of fluid will be at D the middle point of its axis, and that the line AD will be vertical. PAR is the trace of the vertical plane AGFB on the horizontal plane through A.

Let $AO = r$, $OE = 2h$, $\alpha = \angle ADO = \tan^{-1}\dfrac{r}{h}$ (a known angle), $\angle LHN = \theta$, and $AK = x$.

Small element at $N = rd\theta dx$; $\therefore$ pressure on it $= g\rho\, rd\theta dx \,.\, RN = g\rho rd\theta dx\left\{x\cos\alpha + (r + r\cos\theta)\sin\alpha\right\}$. Since it acts normally to the

surface, its direction is through H the centre of the section. The resolved part along LK is $= g\rho r d\theta dx \left\{ x\cos\alpha + (r + r\cos\theta)\sin\alpha \right\} \cos\theta$;
therefore the resolved pressure along LK from M and N is

$$= 2g\rho r d\theta dx \left\{ x\cos\alpha + (r + r\cos\theta)\sin\alpha \right\} \cos\theta.$$

The resolved pressures perpendicular to KL are equal and opposite, and therefore destroy each other.

Therefore the pressure (perpendicular to FB) on the whole curved surface

$$= 2g\rho \int_0^{2h}\int_0^{\pi} dx \left\{ x\cos\alpha\cos\theta\, d\theta + r\sin\alpha\cos\theta\, d\theta + r\sin\alpha\cos^2\theta\, d\theta \right\}$$

$$= 2g\rho r \int_0^{2h} dx \,.\, \frac{\pi}{2} r\sin\alpha = 2g\rho\pi r^2 h \sin\alpha$$

= the resultant pressure on the curved surface.

The axis of the cylinder makes an angle with the horizontal $= \frac{1}{2}\pi - \alpha =$ the angle LK makes with AD, which proves the proposition in the question.

[We add here a solution of an analogous question which was proposed for investigation in our review of Besant's *Hydrostatics*, in the *Educational Times* for May, 1864.

A hollow cone without weight, closed and filled with water, is suspended from a point in the rim of its base; prove that (1), if α be the semivertical angle of the cone, the total pressures on the curved surface and the base are in the ratio $(1+11\sin^2\alpha) : 12\sin^3\alpha$; and (2), if ϕ be the angle which the direction of the resultant pressure on the curved surface makes with the vertical, $\cot\phi = \frac{1}{48}(\cot^3\alpha + 28\cot\alpha)$.

1. Let B, C be the respective areas of the base and curved surface of the cone; b, c the depths of their centres of gravity below the point of suspension; P_b, P_c the whole pressures of the fluid on these areas; and w the weight of a unit of volume; then $P_b = Bbw$, and $P_c = Ccw$. Now the centres of gravity of the volume and surface of the cone are at the respective distances of three-fourths and two-thirds of the axis from the vertex; also the point of suspension and the centre of gravity of the volume of the cone must be in the same vertical line; hence, putting a for the radius of the base of the cone, and β for the inclination of the axis to the vertical, we have $B = \pi a^2$, and $C = \pi a^2 \operatorname{cosec}\alpha$; also

$a\cot\beta = \frac{1}{4}a\cot\alpha$, or $\tan\beta = 4\tan\alpha$; $b = a\sin\beta$; $c = a(\sin\beta + \frac{1}{3}\cot\alpha\cos\beta)$;

$\therefore\ P_b : P_c = b : c\operatorname{cosec}\alpha = \sin\alpha : 1 + \frac{1}{3}\cot\alpha\cot\beta = 12\sin^3\alpha : 1 + 11\sin^2\alpha.$

2. The resultant vertical pressure on the whole surface is equal to the weight of the fluid, that is to $\frac{1}{3}\pi a^3 w\cot\alpha$; also the vertical component of the whole pressure on the base is $P_b\cos\beta = \pi a^3 w\sin\beta\cos\beta$; therefore the resultant vertical pressure (V) on the curved surface is

$$V = \pi a^3 w\left(\tfrac{1}{3}\cot\alpha + \sin\beta\cos\beta\right).$$

The horizontal pressure on the curved surface is equal to the horizontal component (H) of the whole pressure on the base; therefore

$$H = \pi a^3 w \sin^2\beta.$$

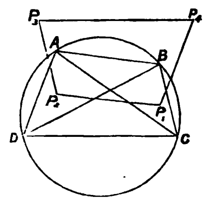
Hence $\cot\phi = \dfrac{V}{H} = \frac{1}{3}\cot\alpha\,(1 + \cot^2\beta) + \cot\beta$

$$= \tfrac{1}{3}\cot\alpha\left(1 + \tfrac{1}{16}\cot^2\alpha\right) + \tfrac{1}{4}\cot\alpha = \tfrac{1}{48}(\cot^3\alpha + 28\cot\alpha).]$$

1986. (Proposed by J. GRIFFITHS, M.A.)—Given four points on a circle; it is required to show that the "polar centres" of the four triangles that can be formed from them lie on another circle of equal radius.

Solution by J. DALE; S. W. BROMFIELD; *and others.*

Taking the "polar centre" of a triangle to signify the centre of the circle with respect to which the triangle is self-conjugate, then the "polar centre" is identical with the intersection of the perpendiculars.



Let A, B, C, D be the four points, and P_1, P_2, P_3, P_4 the polar centres of the triangles BCD, CDA, DAB, ABC respectively. Then, as AP_2 and BP_1 are perpendicular to DC and equal to each other, therefore P_1P_2 is equal and parallel to AB, so also P_2P_3, P_3P_4, P_4P_1 are respectively equal and parallel to BC, CD, DA.

Hence the quadrilateral $P_1P_2P_3P_4$ is equal in all respects to the quadrilateral ABCD, and consequently the circle through $P_1P_2P_3P_4$ is equal to the circle through ABCD.

NOTE.—It may be readily shown that the lines AP_1, BP_2, CP_3, DP_4 meet in one point, and hence that the quadrilateral $P_1P_2P_3P_4$ is obtained by turning ABCD through an angle of 180° round this point.

1844. (Proposed by W. H. LAVERTY.)—If $(\epsilon)_1$ represent ϵ^x, $(\epsilon)_2$ represent ϵ^x, &c.; and if $\log^2(x)$ represent $\log_\epsilon(\log_\epsilon x)$, &c.; find the value of

$$\int_0^\infty \frac{(\epsilon)_{n-1}\cdot(\epsilon)_{n-2}\cdots(\epsilon)_1\,.\,dx}{(\epsilon)_n\cdot\sqrt{(\epsilon)_{n-1}}};\quad \int_0^\infty \frac{\log^n(x)\,.\,dx}{\left\{\log^{n-1}(x)\right\}^2.\log^{n-2}(x)\ldots\log(x).x}$$

Solution by the PROPOSER.

Let $(\epsilon)_{n-1}=y$; then $(\epsilon)_n=\epsilon^y$, and $(\epsilon)_{n-1}\cdot(\epsilon)_{n-2}\cdots(\epsilon)_1\,.\,dx=dy$; therefore the first integral is equal to

$$\int^\infty \frac{dy}{(\epsilon)_n\cdot\sqrt{(\epsilon)_{n-1}}}=\int^\infty \epsilon^{-y}y^{-\frac{1}{2}}\,dy=\Gamma(\tfrac{1}{2})=\sqrt{(\pi)}.$$

2. Let $\log^n(x)=y$; then $\log^{n-1}(x)=\epsilon^y$, and

$$\frac{dx}{\log^{n-1}x).\log^{n-2}(x)\ldots\log(x).x}=dy;$$

therefore the second integral is equal to

$$\int_0^\infty \epsilon^{-y}y\,dy=\Gamma(2)=1.$$

THEOREM: BY PROFESSOR CAYLEY.

If (A, A′), (B, B′) are four points (two real and the other two imaginary) related to each other as foci and antifoci, (that is, if the lines AA′, BB′ intersect at right angles in a point O in such wise that $OA = OA' = i.OB = i.OB'$), then the product of the distances of any point P from the points A, A′ is equal to the product of the distances of the same point P from the two points B, B′.

In fact, the coordinates of A, A′ may be taken to be $(a, 0)$, $(-a, 0)$, and those of B, B′ to be $(0, ai)$, $(0, -ai)$; whence, if (x, y) are the coordinates of P, we have

$$(AP)^2 = (x-a)^2+y^2 = (x-a+iy)(x-a-iy)$$
$$(A'P)^2 = (x+a)^2+y^2 = (x+a+iy)(x+a-iy)$$
$$(BP)^2 = x^2+(y-ia)^2 = (x+iy+a)(x-iy-a)$$
$$(B'P)^2 = x^2+(y+ia)^2 = (x+iy-a)(x-iy+a),$$

from which the theorem is at once seen to be true.

An important application of the theorem consists in the means which it affords of passing from the foci (A, B, C, D) of a bicircular quartic, to the antifoci (A, B) and (C, D); viz., if these are (A′, B′, C′, D′), then the equation $l\sqrt{(A)}+m\sqrt{(B)}+n\sqrt{(C)} = 0$, must be transformable into $l'\sqrt{(A')}+m'\sqrt{(B')}+n'\sqrt{(C')} = 0$. Writing these respectively under the forms

$$l^2A+m^2B-n^2C+2lm\sqrt{(AB)} = 0, \quad l'^2A'+m'^2B'-n'^2C'+2l'm'\sqrt{(A'B')}=0,$$

the two radicals $\sqrt{(AB)}$, $\sqrt{(A'B')}$ are identical; and the remaining terms in the two equations respectively are rational functions, which when the ratios $l' : m' : n'$ are properly determined will be to each other in the ratio $lm : l'm'$; the two equations being thus identical.

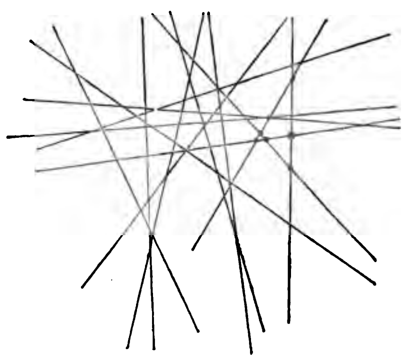
NOTE ON QUESTION 1894. BY W. S. B. WOOLHOUSE, F.R.A.S.

On p. 72 of this volume of the *Reprint* there is a paper by Mr. GODFRAY, "On some Problems in the Theory of Chances." His remarks do not anticipate the statements I have yet to make on the various solutions to Professor SYLVESTER's problem of four points. Mr. GODFRAY, by way of illustration, has referred to my solution to the general problem, Quest. 1894 (*Reprint*, Vol. V., p. 110), and has ventured to give three separate and distinct meanings of "a chord drawn at random." As this may be calculated to mislead, I must really set the matter right. Only one legitimate meaning can attach to the phrase in question, and at the outset of the solution referred to I have proved that a random chord may indiscriminately and with equal probability enter into all the combinations stated in Mr. GODFRAY's Case (1). This, therefore, is not an arbitrary hypothesis, but a matter of fact inherent in the investigation. The Cases (2) and (3) adduced by Mr. GODFRAY are however quite artificial assumptions, which may do very well for the manufacture of mathematical exercises, but they cannot be accepted as correct definitions, or such as are in any way consistent with our elementary notions of the simple meaning of a random chord. There can be no doubt whatever that the only true meaning is that which is adopted in my solution.

On the Four-point Problem. By J. M. Wilson, M.A., F.G.S.

Mr. Godfray's remarks (*Reprint*, Vol. VI., p. 72), tempt me to contribute a few words towards the discussion of the solutions of this problem.

The solution of the problem requires (1) that the definition of a random point be exact and correct, (2) that it be correctly argued from. Now there can be no doubt that the correct definition of a random point is that it is one equally likely to fall in any one of the small equal areas into which we suppose space to be divided. Dr. Ingleby's definition (*Reprint*, Vol. V., p. 82,) is perfectly correct. But Mr. Woolhouse's remarks (*Reprint*, Vol. VI., p. 49) show very clearly that the solution is wrong; because it implies that the triangle formed by the first three points is finite, while the fourth point, and the fourth point only, has an infinite range. In fact δ is not infinitely small compared with $\alpha+\beta+\gamma$, the assumption on which his solution depends. My own solution is of a different kind. I will here reproduce it in an altered form.



Draw a number of intersecting lines as in the figure, at random, that is, according to no law (for to speak of random lines drawn according to a special law, with Mr. Godfray, is to me unintelligible). They determine by their intersection a determinate number of points. These points may be grouped in fours; (1) completely, taking every four; (2) partially, taking fours which are all formed by four lines; excluding in both cases the groups in which more than two lie on the same line.

The first method is not a basis for calculation, so far as I can see.

Adopting the second, I observe that, grouping the lines by fours, I get for every four lines three groups of four points, in which one group always forms a reentrant quadrilateral, two groups always convex ones. This enables me to say, that of all the groups of four points which lie on four lines in the figure, one-third form reentrant quadrilaterals. Or, in other words, if a finite number of lines be drawn on a sheet of paper large enough to admit of their intersection, and four points be taken at random among the points of their intersection, such that the four always lie on four lines, the probability of the four forming a reentrant quadrilateral is $\frac{1}{3}$. Up to this point I imagine there can be no dispute. To proceed. Let the number of the lines be increased very greatly. The points of intersection will spread uniformly over space, that is to say, in equal areas there will ultimately be equal numbers of them. And this will be equally true whether we take the complete or partial grouping of the points. Hence the result $\frac{1}{3}$ is true for any finite number of lines, and therefore for an infinite number of lines; and *when the number of lines is infinite the points will satisfy the definition of random points*, that is, they will be equally distributed over space.

I may add, that when the four points are taken within any triangle, the result has been independently and unquestionably shown to be $\frac{1}{3}$.

NOTE ON THE GEOMETRY OF THE TRIANGLE. BY J. GRIFFITHS, M.A.

If P denote the point of intersection of the three perpendiculars, and G the centroid of any triangle ABC; then, as we know, the following circles are coaxal; viz., 1, The nine-point circle, the circumscribing and self-conjugate circles; 2, The circle described upon PG as diameter (see *Reprint*, Vol. II., p. 25). The object of the present Note is to point out another circle of the system in question.

Let S denote the circle circumscribing the triangle; S′ the self-conjugate circle; and S″ the circle circumscribing the triangle A′B′C′, formed by the tangents to S at the vertices A, B, C; then the circles S, S′, S″ are coaxal.

For, taking the triangle ABC as triangle of reference, and adopting the ordinary notation, we easily find the equations to the vertices of the triangle A′B′C′ to be $\frac{\alpha}{-a}=\frac{\beta}{b}=\frac{\gamma}{c}$, $\frac{\alpha}{a}=\frac{\beta}{-b}=\frac{\gamma}{c}$, $\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{-c}$;
and that to the circle S″, which passes through these three points, to be

$$a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2+\lambda\,(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=0,$$

where $\lambda=\dfrac{a^3\cos A+b^3\cos B+c^3\cos C}{abc}=1+4\cos A\cos B\cos C.$

Hence the theorem in question. Hence also, if α, β, γ be the points of contact of the sides of the triangle ABC with one of the four circles (I) which can be drawn to touch them; then the nine-point circle of the triangle $\alpha\beta\gamma$, and the circles S and I, are coaxal.

1878. (Proposed by W. K. CLIFFORD.)—A line of length a is broken up into n pieces at random; prove that (1) the chance that they cannot be made into a polygon of n sides is $n2^{1-n}$; and (2) the chance, that the sum of the squares described on them does not exceed

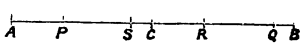
$$\frac{a^2}{n-1},\ \text{is}\quad \left(\frac{\pi}{n^2-n}\right)^{\frac12(n-1)}\frac{\Gamma(n)}{\Gamma\left\{\frac12(n+1)\right\}}\cdot\frac{1}{n^{\frac12}}.$$

Solution by the PROPOSER.

1. Let us define as follows. A point is taken *at random* on a (finite or infinite) straight line, when the chance that the point lies on a finite portion of the line varies as the length of that portion. And, a line is broken up at random when the points of division are taken at random.

Now, the n pieces will always be capable of forming a polygon except when one of them is greater than the sum of all the rest; that is, greater than half the line. The *first* part of the question may therefore be stated thus; $n-1$ points are taken at random on a finite line; to find the chance that some one of the intervals shall be greater than half the line.

2. *First* Solution. Bisect the line AB at C. Then the chance that one of the points of division shall lie within BC is



$\frac12$; therefore the chance that all the $n-1$ points shall lie within BC is 2^{1-n}.

But this is the chance that the *first piece* (reckoning from A) shall be greater than AC. Next, I say that the chance of the rth piece being greater than half the line is equal to the chance of the $(r+1)$th piece being greater. For, let PQ be the portion which is made up of the rth and $(r+1)$th pieces. And take $PR = QS = AC$. Then if the point of division between the rth and $(r+1)$th pieces lies within RQ, the rth piece is greater than AC; and if it lies within PS, the $(r+1)$th piece is greater than AC. But $RQ = PS$; therefore by definition the chances are equal. Consequently, the chance that any one of the n pieces shall be greater than AC is equal to the chance that any other of the n pieces shall be greater than AC. And all these n events are mutually exclusive; while we have proved that the chance of the first of them is 2^{1-n}. Therefore the chance that some one piece is greater than AC is $n2^{1-n}$.

3. *Second* Solution. I am convinced that there is a fallacy in the above, and have therefore tried to get a rigorous proof in this way. Take P a point in AC, and let $AP = x$. Consider a small element dx at P. I want to find the chance that the rth piece, reckoning from A, may begin at P (within the element dx) and be greater than AC. This requires, first, that one of the $n-1$ points of division shall be within dx; the chance of this is $(n-1)\dfrac{dx}{a}$; next, $r-2$ of the remaining points must be within AP, and the chance of this is $\dfrac{\lfloor n-2}{\lfloor n-r \lfloor r-a}\left(\dfrac{x}{a}\right)^{r-2}$; lastly, the $n-r+1$ points left must be within RB; whose chance is $\left(\dfrac{1}{2}-\dfrac{x}{a}\right)^{n-r+1}$. Therefore the chance required is $\dfrac{\lfloor n-1}{\lfloor n-r \lfloor r-2}\left(\dfrac{x}{a}\right)^{r-2}\left(\dfrac{1}{2}-\dfrac{x}{a}\right)^{n-r+1}\dfrac{dx}{a}$.

Now, if we integrate this with respect to x from 0 to $\frac{1}{2}a$, we shall get the entire chance that the rth piece may be greater than AC. The integral is easily found to be 2^{1-n}. And as there are thus n equal chances, whose events are all mutually exclusive, the chance that some one of these events will happen is $n2^{1-n}$.

4. *Third* Solution. To make this clear, I will state first the previously known analogous solutions in the cases where $n = 3$ and $n = 4$. When the line is divided into three pieces, call them x, y, z, and take their lengths for the coordinates of a point P in geometry of three dimensions. Then, since

$$x + y + z = a \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and x, y, z are all positive, the point P must be somewhere on the surface of the equilateral triangle determined on the plane (1) by the coordinate planes. Now, consider those points on the triangle for which $x > \frac{1}{2}a$. These are cut off by the plane $x = \frac{1}{2}a$; and it is easy to see that this plane cuts off from one corner of the triangle a similar triangle of *half the linear dimensions*, and therefore of $\frac{1}{4}$ the area. Now, there are three corners cut off; their joint area is therefore $\frac{3}{4}$ of the area of the triangle; and the chance required is accordingly $\frac{3}{4}$.

When the line is divided into *four* pieces, take the *first three* pieces as the coordinates of a point in space. Then we have $x + y + z < a$, and x, y, z all positive; so the point must lie within the content of the tetrahedron bounded by the plane $x + y + z = a$ and the coordinate planes. Now, if $x + y + z < \frac{1}{2}a$, the *fourth piece* must be *greater* than $\frac{1}{2}a$. The points for

which this is the case are cut off by the plane $x+y+z = \frac{1}{2}a$; and it is easily seen as before that this plane cuts off from one corner of the tetrahedron a similar tetrahedron of half the linear dimensions, and therefore of $\frac{1}{8}$ the volume. So also the plane $x = \frac{1}{2}a$ cuts off from another corner a similar tetrahedron of half the linear dimensions. Since therefore there are four corners cut off, their joint volume is $\frac{4}{8}$ or $\frac{1}{2}$ of the volume of the tetrahedron; and the chance required is accordingly $\frac{1}{2}$.

5. Now, consider the analogous cases in geometry of n dimensions. Corresponding to a closed area, and a closed volume, we have something which I shall call a *confine*. Corresponding to a triangle, and to a tetrahedron, there is a confine with $n+1$ corners or vertices, which I shall call a *prime* confine, as being the simplest form of confine. A prime confine has also $n+1$ faces, each of which is, not a plane, but a prime confine of $n-1$ dimensions. Any two vertices may be joined by a straight line, which is an *edge* of the confine. Through each vertex pass n edges. A prime confine may be *regular*, which it is when any three vertices form an equilateral triangle; or *rectangular*, which it is when the edges through some one vertex are all equal and at right angles to one another.

To solve the question for general values of n, we may adopt as a type either of the geometrical solutions given for the cases $n = 3$ and $n = 4$. First, take the lengths of the n pieces for the coordinates of a point in geometry of n dimensions. Then, since their sum is a, and they are all positive, the point must lie within a certain regular prime confine of $n-1$ dimensions. The supposition that a certain piece is greater than $\frac{1}{2}a$ cuts off from one corner of the confine a similar confine of half the linear dimensions, and therefore of 2^{1-n} times the content. And as there are n corners, their joint content is $n\,2^{1-n}$ times the content of the confine; the chance required is consequently $n\,2^{1-n}$. Or, take the lengths of the *first* $n-1$ *pieces* as the coordinates of a point in geometry of $n-1$ dimensions; the point will then lie within a certain *rectangular* confine of $n-1$ dimensions; and the investigation proceeds as before, the n corners being cut off in the same manner.

6. It will be seen that this *third* solution involves in a geometrical form the assumption of which some sort of proof was given in the *first* solution. Let us make this extension of our fundamental definition:—A point is taken *at random* in a (finite or infinite) space of n dimensions, when the chance that the point lies in a finite portion of the space varies as the content of that portion. The assumption is that when the lengths of the pieces into which a line is broken up are taken as coordinates of a point, then if the line is broken up at random the point is taken at random, and *vice versâ*. The proof of this assumption may be shown to involve a geometrical proposition equivalent to the integration by parts of the differential in Art. (3).

Making this assumption, we may solve the second part of the question by the method of the *third* solution of the first part. I will first state the previously known analogous solution of the case where $n = 3$. The question is in this case,—If a line of length a be broken into three pieces at random, find the chance that the sum of the squares of these pieces shall be less than $\frac{1}{2}a^2$. Take the lengths of the three pieces for coordinates x, y, z of a point P in geometry of three dimensions; then, as before, the point must lie somewhere in the area of the equilateral triangle determined on the plane $x+y+z = a$ by the coordinate planes. But if also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain circle determined on the plane $x+y+z = a$ by the sphere $x^2+y^2+z^2 = m^2$. Now, in the case where $m^2 = \frac{1}{2}a^2$ this circle is the circle

inscribed in the equilateral triangle; so that the question reduces itself to this one:—

To find, in terms of the area of an equilateral triangle, the area of its inscribed circle.

Now let us go a little further, and consider the case in which $n = 4$. Here we shall have to take a point P in geometry of four dimensions; the point must lie somewhere in the regular tetrahedron determined on the hyper-plane $x+y+z+w = a$ by the coordinate hyper-planes. If also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain sphere determined on the hyper-plane $x+y+z+w = a$ by the quasi-sphere $x^2+y^2+z^2+w^2 = m^2$. In the particular case where m is the perpendicular from the vertex on the base of a rectangular tetrahedron each of whose equal edges is of length a, or $m^2 = \frac{1}{3}a^2$, this sphere is the sphere inscribed in the regular tetrahedron. The question is therefore reduced to this one:—

To find, in terms of the volume of a regular tetrahedron, the volume of its inscribed sphere.

Now, a similar reduction holds in the general case; viz., the question can always be reduced to this one:—

To find, in terms of the content of a regular prime confine of $n-1$ dimensions, the content of its inscribed quasi-sphere.

This question I proceed to solve.

7. Let $n-1 = p$. *The perpendicular from any vertex on the opposite face of a regular prime confine in p dimensions* $= \left(\dfrac{p+1}{2p}\right)^{\frac{1}{2}}$. (edge).

For, let O be the vertex in question, OA, OB,.... the p edges through O. Draw through each vertex A a space of $p-1$ dimensions parallel to the face opposite to A. The p spaces thus drawn will intersect in a point P, such that OP is the diagonal of a confine analogous to a parallelogram and to a parallelepiped. Then OP is p times the perpendicular from O on the opposite face of the regular confine; for the perpendicular is the projection of *one* edge at a certain angle, while OP is the projection at the same angle of a broken line consisting of p edges.

We have also

$$\begin{aligned} OP^2 &= OA^2+OB^2+OC^2+\ldots . +2OA \cdot OB \cos AOB + \ldots . \\ &= \Sigma . OA^2 + \Sigma . OA . OB, \ [\text{since} \cos AOB = \tfrac{1}{2}, \ \&c.], \\ &= \left\{p+\tfrac{1}{2}p(p-1)\right\} . OA^2 = \tfrac{1}{2}p(p+1) . OA^2, \end{aligned}$$

therefore $(\text{perpendicular})^2 = \dfrac{OP^2}{p^2} = \dfrac{p+1}{2p} . (\text{edge})^2.$

[If the confine were *rectangular*, or all the angles at O right angles, we should have cos AOB = 0, &c.: and so

$$(\text{perpendicular})^2 = \frac{1}{p}\,(\text{edge})^2 = \frac{a^2}{n-1};$$

which proves that the question *does* always reduce itself to the one now under consideration.]

The content *of a regular prime confine in p dimensions whose edge is a,* is

$$= \frac{a^p}{\underline{|p}} \left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}$$

Suppose this formula true for $p-1$ dimensions; that is, let

$$V_{p-1} = \frac{a^{p-1}}{\underline{|p-1}} \cdot \left(\frac{p}{2^{p-1}}\right)^{\frac{1}{2}}.$$

Now, content of confine $= \dfrac{1}{p} \times$ perpendicular $\times$ content of face, or

$$V_p = \frac{a}{p} \cdot \left(\frac{p+1}{2^p}\right)^{\frac{1}{2}} \cdot V_{p-1} = \frac{a^p}{\underline{|p}} \cdot \left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}.$$

Hence the formula, if true for one value of p, is true for the next; now it can be immediately verified in the case of $p = 1$; therefore it is generally true.

The radius of the inscribed quasi-sphere $= \dfrac{a}{\left\{2p\,(p+1)\right\}^{\frac{1}{2}}}$.

We can divide the regular confine into $p+1$ equal confines, each having the centre of the inscribed quasi-sphere for vertex; and the content of one of these $= \dfrac{\rho}{p} \times$ content of face; but the sum of them all is equal to the content of the whole confine. Hence $(p+1)\,\rho =$ perpendicular of confine

$$= a\left(\frac{p+1}{2p}\right)^{\frac{1}{2}}, \quad \text{or, } \rho = \frac{a}{\left\{2p\,(p+1)\right\}^{\frac{1}{2}}}.$$

The content of the quasi-sphere $= \rho^p \dfrac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}}{\Gamma\left(\frac{1}{2}p+1\right)}$.

For it is the value of $\iiint \ldots dx\, dy\, dz \ldots$ the integral being so taken as to give to the variables all values consistent with the condition that $x^2 + y^2 + z^2 + \ldots$ is not greater than ρ^2. (See TODHUNTER'S *Integral Calculus*, Art. 271.) Let C_p denote this content; then

$$C_p = \rho^p \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^p}{\Gamma\left(\frac{1}{2}p+1\right)} = \frac{a^p}{(2p^2+2p)^{\frac{1}{2}p}} \cdot \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^p}{\Gamma\left(\frac{1}{2}p+1\right)}$$

therefore $$\frac{C_p}{V_p} = \left(\frac{\pi}{p^2+p}\right)^{\frac{1}{2}p} \cdot \frac{\Gamma(p-1)}{\Gamma\left(\frac{1}{2}p+1\right)} \cdot \frac{1}{(p+1)^{\frac{1}{2}}}.$$

Restore $n-1$ for p, and we get the answer to the question, namely,

$$\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)} \cdot \frac{\Gamma(n)}{\Gamma\left\{\frac{1}{2}(n+1)\right\}} \cdot \frac{1}{n^{\frac{1}{2}}}.$$

8. The following are applications of the same method.

If a line be broken up at random into n pieces, the chance of an assigned two of them (the pth and qth from one end) being together greater than half the line, is $n\,2^{1-n}$.

If n pieces be cut off at random, one from each of n equal lines, the chance that the pieces cannot be made into a polygon is $\dfrac{1}{\underline{|n-1}}$.

1990. (Proposed by Professor SYLVESTER.)—

(1.) Prove that the locus of one set of foci of all the conics that touch a given circle at two given points, is another circle passing through those points and the centre of the given circle.

(2.) Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

(3.) Prove that a circular cubic is the locus of one set of foci of all the conics that can be drawn through four points lying in a circle.

(4.) Prove that, if a circle and straight line be cut by any transversal in three points, these will be the foci of one of a system of Cartesian ovals having double contact with one another at two fixed points. [This last proposition is Mr. CROFTON'S, and may be proved as a particular case of (2)].

Solution by PROFESSOR HIRST.

The following remarks have reference solely to the first and third parts of the Question, and to the mode of transition from the latter to the former.

Let 1, 2, 3, 4 be any four points on a circle whose centre is O, and let α, β, γ be the intersections of the three pairs of opposite sides of the quadrangle which they form. Or more precisely, let

$\overline{41}$ and $\overline{23}$ (or A and A′) intersect in α,

$\overline{42}$ and $\overline{31}$ (or B and B′) intersect in β,

$\overline{43}$ and $\overline{12}$ (or C and C′) intersect in γ.

Now, by the theorem of Desargues, the line-pairs A, A′; B, B′; C, C′ cut the line at infinity in three pairs of points in involution; and to this involution belong the intersections o and o' of the circle (O) (the circular points at infinity), as well as the infinitely distant points s and s' of every conic (S) circumscribed to the quadrangle 1 2 3 4. The double points δ and $\delta_{,}$ of this involution divide every segment thereof harmonically; hence, S being the centre of (S), the angle between its asymptotes Ss, Ss' is divided harmonically by $S\delta$ and $S\delta_{,}$, and the latter are accordingly conjugate diameters of (S). Moreover, δ and δ' being harmonic conjugates relative to o and o', $S\delta$ and $S\delta_{,}$ are at right angles to each other; consequently they are the axes of (S), and we conclude that *the axes of every conic* (S) *circumscribed to a quadrangle* 1 2 3 4 *which is itself inscribed in a circle* (O) *are parallel to two fixed orthogonal lines* $O\delta$, $O\delta_{,}$.

The line-pairs A, A′; B, B′; C, C′ being included amongst the conics of the system to which (S) belongs, we may at once infer that the bisectors of the adjacent angles formed by any one of these pairs are also parallel to the fixed lines $O\delta$, $O\delta_{,}$. (See Professor SYLVESTER'S Question 1950, *Reprint*, Vol. V., p. 105.)

In the system of circumscribed conics, there are two parabolas (P) and ($P_{,}$) (they are the conics which touch the line at infinity in the double points δ and $\delta_{,}$), and since these are the only conics of the system which have infinitely distant centres, we conclude that the *locus of the centres of all conics is an equilateral hyperbola* (H) *whose asymptotes are parallel to* $O\delta$ *and* $O\delta_{,}$. (H) passes obviously through the centres α, β, γ, and O of the line-pairs A, A′; B, B′; C, C′; and of the circle (O).

Let us now consider any line directed towards the infinitely distant double point δ. Exclusive of the parabola (P) there is but one conic (S) of the system which has its centre thereon, since this line is cut in only one point S, at a finite distance, by the hyperbola (H). Consequently, exclusive of δ, which

must be regarded as a focus of the parabola (P), there are but two *conjugate* foci $f, f_,$ of the conics of our system which are situated on any line Sδ parallel to one system of axes. Hence *the locus of such foci is a cubic* (Σ) *which has one asymptote parallel to* Oδ. This asymptote, in fact, is also an asymptote of the hyperbola (H), for when Sδ coincides with the latter, S and one of the foci $f_,$ recede to infinity. Moreover *this cubic* (Σ) *is circular*, for when Sδ moves parallel to itself to infinity, the conic (S) which has its centre thereon is the parabola ($P_,$) whose conjugate imaginary foci are the circular points o and o'. Lastly, the *circular cubic* (Σ) *passes obviously through the points* α, β, γ, O, and since in each of these points two conjugate foci $f, f_,$, on diameters directed towards δ, coincide, *the tangents to the cubic* (Σ) *at the points* α, β, γ, O *are parallel to the fixed line* Oδ.

In like manner *the locus of the conjugate foci situated on axes parallel to* O$\delta_,$ *is a circular cubic* ($\Sigma_,$), *whose real asymptote coincides with the asymptote of the hyperbola of centres* (H) *which is directed towards* $\delta_,$, *and whose tangents at the points* α, β, γ, O *are parallel to this asymptote*.

We are now in a position to pass to the first part of our Question, and to examine what becomes of the cubics (Σ) and ($\Sigma_,$) when the four points 1, 2, 3, 4 coincide two and two, say in a and a'. In making this transition we may clearly suppose α, β, and γ, which are the angles of a self-conjugate triangle relative to (O), to remain fixed. Let then 1 and 4, as well as 2 and 3, be made to approach so as ultimately to coincide respectively with a and a' on the polar $\beta\gamma$ of α. The line-pair A, A$'$ will become a pair of tangents αa, $\alpha a'$ and B, B$'$ and C, C$'$ (which are still supposed to intersect in β and γ respectively) will ultimately coincide upon the chord of contact aa'. The points δ and $\delta_,$ are now the infinitely distant points of αO and aa' and the lines αOδ and $aa'\delta_,$ must be constituent parts of the cubics (Σ) and ($\Sigma_,$), respectively; for on each of these lines, say αOδ, five points of the cubic are situated, viz., two coincident in α, two others in O, and a fifth in δ. The remaining constituent of each circular cubic must of course be a circle. In the case of the cubic (Σ) this circle must pass through β and γ and have tangents at these points parallel to αO. Consequently (Σ) *must break up into the line* αO *and the circle whose diameter is* $\overline{\beta\gamma}$. Similarly ($\Sigma_,$) *must break up into the line* $\beta\gamma$ *and the circle whose diameter is* $\overline{\alpha \mathrm{O}}$, which circle manifestly passes through a and a'.

It should be observed, however, that β and γ are indeterminate so long as the *manner* in which the points 1 and 4, and 2 and 3, approach coincidence is not stated. Consequently (Σ) may be said to resolve itself into the line αO and an *indeterminate circle* having its centre on the line aa' and cutting (O) orthogonally; the reason of this being that any two points whatever equidistant from a line may be regarded as the foci of a conic which consists of a pair of *unlimited* lines coincident therewith. But if we exclude the foci of the doubled line through a and a', regarded as such an exceptional conic, we may disregard the indeterminate constituent circle ($\beta\gamma$) of (Σ) as well as the constituent line $\beta\gamma$ of ($\Sigma_,$), and say that *the foci of a system of conics having double contact at* a *and* a' *with a circle* (O), *lie on the circular cubic consisting of the diameter of* (O) *which bisects the chord* aa', *and of the circle whose diameter is the intercept between the pole of* aa', *and the centre of* (O).

In conclusion I may remark that we should have arrived directly at this result by the method of CHASLES, for the characteristics of the present system are both equal to unity. (See my Solution of Question 1717, *Reprint*, Vol. IV., p. 19.)

1853. (Proposed by J. WILSON.) — Find two series of integral cubes such that every term in the first may be the sum, and every term in the second the difference, of two integral squares. Also find two series of integral squares, such that every term in the first may be the sum, and every term in the second the difference, of two integral cubes.

Solution by SAMUEL BILLS.

First; to find cube numbers which shall be the sum or difference of two squares, assume the equation $x^3 = p^2x^2 \pm q^2x^2$, then $x = p^2 \pm q^2$, where p and q may be taken at pleasure. This answers the first part of the question.

Again; to find square numbers which shall be the sum or the difference of two cubes, assume the equation $a^2x^2 = p^3x^3 \pm q^3x^3$, then $x = \dfrac{a^2}{p^3 \pm q^3}$, where a, p and q may be taken at pleasure; which satisfies the second part of the question. It is evident that an indefinite number of answers may be found in each case. To obtain integers, a^2 must be divisible by $p^3 \pm q^3$.

2240. (Proposed by the Rev. R. H. WRIGHT, M.A.)—If a conic be described about a triangle ABC, and tangents at (B, C), (C, A), (A, B) meet respectively in G, H, K; then, if D, E, F be any three points in BC, CA, AB such that AD, BE, CF are concurrent, the three lines GD, HE, KF will also be concurrent.

I. *Solution by* S. WATSON; W. CHADWICK; W. H. LAVERTY; J. DALE; *the* PROPOSER; *and others.*

Let the equation of the conic be $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, then the equations of the tangents at A, B, C are respectively

$$\frac{\beta}{m} + \frac{\gamma}{n} = 0, \quad \frac{\gamma}{n} + \frac{\alpha}{l} = 0, \quad \frac{\alpha}{l} + \frac{\beta}{m} = 0 \ldots\ldots (1, 2, 3);$$

also the equations of AD, BE, CF are all comprehended in

$$p\alpha = q\beta = r\gamma \ldots\ldots\ldots\ldots\ldots (4).$$

From the above equations, those of the three lines GD, HE, KF are easily found to be all comprehended in

$$rp\,(l\gamma + n\alpha) = pq\,(m\alpha + l\beta) = qr\,(n\beta + m\gamma) \ldots\ldots (5);$$

hence GD, HE, KF meet in the point determined by (5).

II. *Solution by* ARCHER STANLEY; W. CHADWICK; *and others.*

It is manifest that the equations

$$\frac{\sin \mathrm{DGB}}{\sin \mathrm{DGC}} = \frac{\mathrm{DB}}{\mathrm{DC}} : \frac{\mathrm{BG}}{\mathrm{CG}}, \quad \frac{\sin \mathrm{EHC}}{\sin \mathrm{EHA}} = \frac{\mathrm{EC}}{\mathrm{EA}} : \frac{\mathrm{CH}}{\mathrm{AH}}, \quad \frac{\sin \mathrm{FKA}}{\sin \mathrm{FKB}} = \frac{\mathrm{FA}}{\mathrm{FB}} : \frac{\mathrm{AK}}{\mathrm{BK}}$$

are true *numerically*, wherever D, E and F may be situated on BC, CA and AB.

But A, B, C being the points of contact of a conic inscribed in GHK, it

follows from a well known modification of Brianchon's theorem that GA, HB and KC are concurrent, and hence, by Ceva's theorem, that unity is the numerical value of the ratio $\dfrac{BG \,.\, CH \,.\, AK}{CG \,.\, AH \,.\, BK}$.

The multiplication of the preceding equations therefore gives

$$\frac{\sin DGB \,.\, \sin EHC \,.\, \sin FKA}{\sin DGC \,.\, \sin EHA \,.\, \sin FKB} = \frac{DB \,.\, EC \,.\, FA}{DC \,.\, EA \,.\, FB},$$

which is obviously true in sign as well as in magnitude. By Ceva's theorem and its converse therefore we conclude that *whenever* AD, BE, *and* CF *are concurrent*, GD, HE *and* KF *are so likewise.* Moreover by the theorem of Menelaus we conclude that *whenever* D, E *and* F *are collinear*, GD, HE *and* KF *intersect the sides of* GHK *in three collinear points.*

To find the form of all possible integral Solutions of $a^x \pm 1 \searrow x$; where the symbol $\searrow$ denotes "divisible by."

By Morgan Jenkins, B.A.

Let $N \searrow p\,[h^q]$ signify that N is divisible by h^q, but by no higher power of h than that which consists of the product of h^q into the highest power of h contained in p (h being prime). Also let $N \searrow p\,[u]$ denote a corresponding relation for every prime factor of u. In both the proposed cases x must of course be prime to "a."

With respect to $a^x - 1 \searrow x$, we have

(1.) x cannot be prime to $a-1$. For let h be the least prime factor of x, and α the least integer consistent with $a^\alpha - 1 \searrow h$, then $\alpha = (h-1)$ or some measure of $(h-1)$. Therefore, if $h = 2$, $\alpha = 1$. But, if $h > 2$, α cannot be >1; for otherwise, since $a^x - 1 \searrow h$, x would contain $\alpha\,(>1)$ a measure of $(h-1)$, which is contrary to the hypothesis that h is the least prime factor of x.

Therefore $\alpha = 1$, and h is a measure of, and therefore x is not prime to, $a-1$.

Cor.—$2^x - 1 \searrow x$ is impossible, and $3^x - 1 \searrow x$ is impossible, unless x is even (excluding $x = 1$ in both cases).

(2.) If $a^{py} - 1 \searrow y$, y can not be prime to $a^p - 1$. This follows from (1), since $a^{py} - 1 = (a^p)^y - 1$.

(3.) If p be any integer and h a prime number, such that

$$a^p - 1 \searrow h, \text{ say } \searrow [h^q] \text{ then } a^{ph^r} - 1 \searrow [h^{q+r}] \searrow (a^p - 1)\,[h^r].$$

For, let $a^p = 1 + mh^q + \text{multiple of } h^{q+1}$; where m is prime to h.

$$a^{ph} = (1 + mh^q + \text{multiple } h^{q+1})^h = 1 + mh^{q+1} + \text{multiple } h^{q+2},$$

therefore $\qquad a^{ph} - 1 \searrow [h^{q+1}]$, therefore $a^{ph^2} - 1 \searrow [h^{q+2}]$,

and by induction $\qquad a^{ph^r} - 1 \searrow [h^{q+r}] \searrow (a^p - 1)\,[h^r]$.

(4.) These considerations are sufficient to furnish all the solutions of $a^x - 1 \searrow x$. By (1) x cannot be prime to $a-1$. By (3), x may contain any powers of any one or more of the prime factors of $(a-1)$, that is, x may be any measure of any power of $(a-1)$, for if $h, k, \ldots.$ be any or all of the prime factors of $(a-1)$ and $x = h^r k^s \ldots.$

$$a^x - 1 \searrow (a-1)\,[h^r], \quad \text{and also} \searrow (a-1)\,[k^s] \text{ \&c.}$$

therefore $\searrow (a-1)\,[h^r]\,[k^s]\ldots.$, therefore $\searrow (a-1)\,[x]$.

Again, x may contain other factors prime to $(a-1)$. For, let $x = yu_{\prime}$ where $u_{\prime}$ is a measure of a power of $(a-1)$: then, since $a^{yu_{\prime}} - 1 \searrow yu_{\prime} \searrow y$,

by (2), y can not be prime to $a^{u_{\prime}} - 1$;

by (3), y may be any measure of any power of $(a^{u_{\prime}} - 1)$,

and $\quad a^{yu_{\prime}} - 1 \searrow (a^{u_{\prime}} - 1)\,[y] \searrow (a-1)\,[yu_{\prime}]$.

Though y might contain some factors of $(a-1)$, yet these may be excluded without loss of generality. For, if y contain a factor u' of the same form as $u_{\prime}$; say, $y = y'u'$; then y' measures y, and therefore some power of $(a^{u_{\prime}} - 1)$, and therefore some power of $(a^{u'u_{\prime}} - 1)$, and therefore y' might be obtained by beginning with $u'u_{\prime}$, which is of the same form as $u_{\prime}$, instead of $u_{\prime}$; $\therefore$ y may be considered as prime to $(a-1)$. These factors may be said to be of the second order of formation; and we may proceed to find factors of superior orders of formation *ad infinitum*. But, by (2), x cannot contain any factor not formed in the way just pointed out.

The factors of any order of formation may, as has been shown for a particular case, be taken, without loss of generality, so as to be prime to the factors of all preceding orders.

Let U_r denote $u_1 . u_2 \ldots\ldots\ldots . u_r$; then

The complete integral solution of $a^x - 1 \searrow x$ *is*

$x = U_n \equiv u_1 . u_2 \ldots\ldots u_n$; *where*

u_1 *is any measure of any power of* $(a-1)$.

u_2 *is any measure, prime to* $(a-1)$, *of any power of* $(a^{u_{\prime}} - 1)$.

. . . .

u_n *is any measure, prime to* $(a^{U_{n-2}} - 1)$, *of any power of* $(a^{U_{n-1}} - 1)$.

We also have

$$a^{U_n} - 1 \searrow (a^{U_{n-1}} - 1)\,[u_n],$$

which expression gives the highest powers of factors of the nth order of formation; $\quad a^{U_{n-1}} - 1 \searrow (a^{U_{n-2}} - 1)\,[u_{n-1}]$;

and $a^{U_n} - 1$ cannot contain a higher power of any of the prime factors of u_{n-1} than $(a^{U_{n-1}} - 1)$, because $U_n = u_n U_{n-1}$, and u_n is by supposition prime to U_{n-1};

therefore $\quad a^{U_n} - 1 \searrow (a^{U_{n-2}} - 1)\,[u_{n-1}]$,

and so on $\quad \searrow (a-1)\,[u_{\prime}]$ and also $\searrow U_n$.

If "a" be *odd* $(a-1)$ is even, and therefore $u_{\prime}$ may be even; therefore x may (if $u_{\prime}$ be even) contain any measure of any power of $(a^2 - 1)$; therefore x may contain any measure of any power of $(a+1)$ provided x be even.

In this case, since 2 is contained in both $(a-1)$ and $(a+1)$ we must, in calculating the highest powers of the factors of $a \pm 1$, consider 2 as a factor

of the first order only, or more shortly proceed thus. Let $x = 2x'$,

$$a^{x}-1 = (a^{2})^{x'} - 1 \searrow (a^{2}-1)\,[x'] \searrow \frac{a^{2}-1}{2}\,[x].$$

(5.) The solutions of $a^{x}+1 \searrow x$ may be deduced (with a few modifications) from the previous results. *But x cannot contain 2^{2} or any higher power of* 2. For, in this case, $a^{x}+1$ is even, and therefore a odd. Let $x = 2x'$. Then $a^{x'}$ is odd and $a^{x}+1 = (a^{x'})^{2}+1$ is of the form $(4m^{2}+4m+1)+1$.

Therefore $\quad a^{x}+1 \searrow [2]$ if a be odd, and $a^{x}+1$ is odd if a be even.

(α.) Let x be odd, then in (3) first changing -1 into $+1$, and supposing h to be odd or any prime >2, we have

$$a^{ph^{r}}+1 \searrow (a^{p}+1)\,[h^{r}] \text{ if } a^{p}+1 \searrow h.$$

Since $a^{x}+1 \searrow x$ and x is odd, $a^{x}-1$ is prime to x, therefore x is prime to $a-1$.

But $a^{2x}-1 = (a^{2})^{x}-1 \searrow x$; therefore x is not prime to $(a^{2}-1)$ and x is prime to $(a-1)$, therefore x is not prime to $a+1$. In like manner if x be odd and $a^{px}+1 \searrow x$, x cannot be prime to $a^{p}+1$.

Hence solutions of $a^{x}+1 \searrow x$ can be found, similar to those of $a^{x}-1 \searrow x$, by taking odd factors formed from $(a+1)$ instead of factors formed from $(a-1)$.

(β.) Let x be even $=2x'$ where x' is odd. Then $a^{x}+1 = (a^{2})^{x'}+1$; therefore x' must not be prime to $a^{2}+1$, that is, must be prime to $a^{2}-1$; for any odd prime, which measures $a^{2}-1$ would also measure $a^{2x'}-1$ or $a^{x}-1$, and would therefore be prime to $a^{x}+1$.

Hence, when a is odd, beside the solutions indicated in the previous article (5a) we may obtain a different set of solutions consisting of the product of 2 *into odd factors formed from $a^{2}+1$.*

A few illustrations will show the application of these solutions.

$11^{(3221^{p}\times 3^{q}).(2^{2}\times 5)}-1 \searrow (11^{2^{2}\times 5}-1)\,[3221^{p}]\,[3^{q}] \searrow [3221^{p+1}]\,[3^{q+1}]$,
3221 being a prime factor of $11^{5}-1$, and 3 of $11^{2}-1$, and therefore each of these prime factors of $11^{2^{2}\times 5}-1$,

$$11^{(3221^{p}\times 3^{q})\,(2^{2}\times 5)}-1 \searrow (11-1)\,[2^{2}]\,[5] \searrow [2^{3}]\,[5^{2}],$$

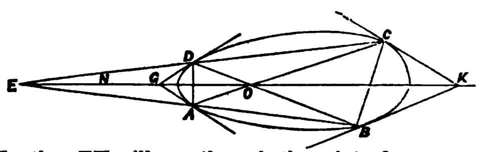
$$5^{7^{s}\times 3}+1 \searrow (5+1)\,[3] \searrow [3^{2}] \searrow (5^{3}+1)\,[7^{s}] \searrow [7^{s+1}],$$

$$5^{13^{s}\times 2}+1 \searrow (5^{2}+1)\,[13^{s}] \searrow [13^{s+1}] \searrow [2].$$

2002. (Proposed by the Rev. R. H. Wright, M.A.)—If a quadrilateral be inscribed in a conic, and either pair of opposite sides BA, CD, be produced to meet in E; then the line joining the point E with G, the intersection of the tangents at A and D, will pass through the intersections of the diagonals of the quadrilateral.

Solution by T. T. WILKINSON, F.R.A.S.; W. CHADWICK; A. COHEN, B.A.; H. TOMLINSON; J. DALE; W. H. LAVERTY; *and many others.*

This question is very neatly solved in Articles 35 and 36 of MACLAURIN'S *General Properties of Geometrical Lines;* and his reasoning may be adapted to the preceding enunciation as follows:—



Let the straight lines AB and CD inscribed in a conic meet in the point E; let straight lines touching the conic in the points B and C meet each other in K: then EK will pass through the point of concourse G of right lines which touch the conic in the points A and D. For if the line EK does not pass through this point of concourse, let it meet one of the lines in G, and the other in N; then since $\frac{1}{EK} \mp \frac{1}{EG} = \frac{1}{EK} \mp \frac{1}{EN}$, we have EG = EN, and the points G and N coincide, contrary to the hypothesis. For similar reasons it appears that AC and DB meet each other in the point O of the same line EK, and therefore the four points E, G, O, K, are in the same straight line. (MACLAURIN'S *Algebra*, pp. 467, 468.) Dr. SIMSON deduces the same theorem from Prop. 49, Book V., of his *Conic Sections;* but there is nothing novel in his mode of solution. M. CHASLES applies his system of Anharmonic Ratios to the same inquiry as follows:— Premising the property that if round two fixed points on a conic two straight lines revolve so as to intersect on the curve; these lines taken in their successive positions will form two homographic pencils; he finds that D and A are two such points; and since AG, DA; AB, DB; AC, DC; AD, DG; intersect on the curve, we have (A.GBCD) = (D.GCBA), and since the two rays AD and DA coincide, the remaining rays intersect in the points E, G, O, which range in the same straight line. Similar considerations will prove that tangents from B and C will intersect in a point K on the same line. Hence the property is fully proved.

1926. (Proposed by A. RENSHAW.)—Find general methods of investigating similar series to Euler's and Machin's used for the calculation of π, and prove the relation

$$\frac{\pi}{4} = \left\{\frac{1}{\pi} - \frac{1}{3\pi^3} + \frac{1}{5\pi^5} - \&c.\right\} + \left\{\frac{\pi-1}{\pi+1} - \frac{(\pi-1)^3}{3(\pi+1)^3} + \&c.\right\}.$$

Solution by the REV. J. L. KITCHIN, M.A.; W. H. LAVERTY; S. W. BROMFIELD; *the* PROPOSER; *and others.*

We have $\tan^{-1}(x_1^{-1}) + \tan^{-1}(x_2^{-1}) + \ldots + \tan^{-1}(x_n)^{-1}$

$$= \tan^{-1}\left\{\frac{x_1^{-1} + x_2^{-1} + \ldots + x_n^{-1} - x_1^{-1}x_2^{-1}x_3^{-1} - \&c. \ldots}{1 - x_1^{-1}x_2^{-1} - x_1^{-1}x_3^{-1} - \&c.}\right\}$$

For Euler's or Machin's series, and for series similar to them, we must have the second side put equal to $\frac{1}{4}\pi$;

$$\text{therefore} \quad \frac{x_1^{-1}+x_2^{-1}+\ldots\ldots+x_n^{-1}-x_1^{-1}x_2^{-1}x_3^{-1}-\&c.\ldots\ldots}{1-x_1^{-1}x_2^{-1}-x_1^{-1}x_3^{-1}-\&c.\ldots\ldots}=1,$$

$$\text{therefore} \quad x_2x_3\ldots\ldots x_n+x_1x_3\ldots\ldots x_n+\ldots\ldots+x_1\ldots\ldots x_n\ldots\ldots-\&c.$$
$$=x_1x_2\ldots\ldots x_n-x_3x_4\ldots\ldots x_n-\&c.\ldots\ldots$$

This equation solved for integer values will give any number of series similar to Euler's and Machin's.

$$\text{From} \qquad \tan^{-1}(x_1^{-1})+\tan^{-1}(x_2^{-1})=\tan^{-1}\left\{\frac{x_1^{-1}+x_2^{-1}}{1-x_1^{-1}x_2^{-1}}\right\}=\frac{\pi}{4},$$

we have $x_1+x_2=x_1x_2-1$. Any solution of this equation will give a series for $\frac{1}{4}\pi$. The only solution in positive integers is that which gives Euler's series, viz., $x_1=2$, $x_2=3$.

$$\text{Again,} \qquad x_2=\frac{x_1+1}{x_1-1};\ \text{put } x_1=\pi,\ \text{then } x_2=\frac{\pi+1}{\pi-1};$$

$$\text{therefore} \qquad \frac{\pi}{4}=\tan^{-1}\left(\frac{1}{\pi}\right)+\tan^{-1}\left(\frac{\pi-1}{\pi+1}\right)$$
$$=\left\{\frac{1}{\pi}-\frac{1}{3\pi^3}+\frac{1}{5\pi^5}-\&c.\right\}+\left\{\frac{\pi-1}{\pi+1}-\frac{(\pi-1)^3}{3(\pi+1)^3}+\&c.\right\}.$$

2006. (Proposed by A. RENSHAW.)—Two conics expressed by their general equations touch one another at the origin; find the condition that they should touch each other in one other point.

Solution by W. CHADWICK; W. H. LAVERTY; J. DALE; E. MCCORMICK; S. W. BROMFIELD; H. TOMLINSON; *the* PROPOSER; *and others.*

Let $S\equiv ax^2+2hxy+by^2+T=0$ be the equation to one conic; then the equation to the other is of the form $S'\equiv a'x^2+2h'xy+b'y^2+T=0$, where T is the common tangent at the origin; therefore if S and S′ touch each other in another point, $S-S'=0$ represents their chord of contact and the condition that

$$(a-a')\,x^2+2\,(h-h')\,xy+(b-b')\,y^2=0$$

should represent two *coincident* straight lines is

$$(h-h')^2=(a-a')\,(b-b'),$$

which is the condition required in the question.

1777. (Proposed by Rev. J. BLISSARD.)—Prove that

$$\Delta^r\log^m(1+\Delta)\,0^n=n\,(n-1)\ldots\ldots(n-m+1)\,\Delta^r\,0^{n-m}.$$

N.B.—From this formula, when r is positive and less than m, by putting $n+m$ for n, we get

$$(n+1)\,(n+2)\ldots\ldots(n+m)\,\Delta^{-r}0^n=\Delta^{m-r}\left(1-\tfrac{1}{2}\Delta+\tfrac{1}{3}\Delta^2-\&c.\right)^m0^{n+m}.$$

Solution by E. FITZGERALD.

From Herschel's theorem (*Examples on Finite Differences*, p. 68),

$$F\epsilon^x = F(1) + F(1+\Delta).0.\frac{x}{1} + F(1+\Delta)0^2.\frac{x^2}{1.2} + \&c. \ldots\ldots (1),$$

therefore $f(\epsilon^x - 1) = f(0) + f(\Delta)0.\frac{x}{1} + f(\Delta)0^2\frac{x^2}{1.2} + \&c.$ .... (2).

Multiply the members of (2) by x^m; then

$$x^m f(\epsilon^x - 1) = x^m f(0) + f(\Delta)0.\frac{x^{m+1}}{1} + f(\Delta)0.\frac{x^{m+2}}{1.2}\&c. \ldots\ldots\ldots (3).$$

But $x^m f(\epsilon^x - 1) = \left\{\log\epsilon^x\right\}^m f(\epsilon^x - 1)$. Expanding this latter by the formula (1) these results

$$\left\{\log\epsilon^x\right\}^m f(\epsilon^x - 1) = 0 + \left\{\log(1+\Delta)\right\}^m f(\Delta)0.\frac{x}{1} + \&c. \ldots\ldots\ldots (4).$$

Equating the coefficients of like powers of x in (3) and (4), we get

$$n(n-1)(n-2)\ldots.(n-m+1)f\Delta 0^{n-m} = \left\{\log(1+\Delta)\right\}^m f(\Delta)0^n \ldots(5).$$

If $f\Delta = \Delta^r$ we have the theorem in question. The theorem (5) is given on p. 310 of De Morgan's splendid work on the Calculus, without demonstration; but this can be easily supplied from the hints there given.

1714. (Proposed by F. D. THOMSON, M.A.)—Investigate the general tangential equation to the foci of a conic, and deduce the general equation to the focus of a parabola and the coordinates of its axis.

Solution by W. S. BURNSIDE, B.A.

The following solution of this problem is contained in Chap. XVIII. of SALMON'S *Conics*.

Let $\Sigma = (A, B, C, F, G, H)\ (\alpha, \beta, \gamma)^2$, $\Omega = \alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma\cos A - 2\gamma\alpha\cos B - 2\alpha\beta\cos C$; then $\Sigma + k\Omega = 0$ will represent points, where k is determined by the equation $\Theta k^2 + \Delta\Theta_1 k + \Delta^2 = 0$. Eliminating k, $\Theta\Sigma^2 - \Delta\Theta_1\Sigma\Omega + \Delta^2\Omega^2 = 0$ is "the general tangential equation of the foci." In the case of a parabola $\Theta = 0$, and this equation becomes $\Theta_1\Sigma - \Delta\Omega = 0$; whence, if (x_1, y_1, z_1) (x_2, y_2, z_2) be the coordinates of the foci, $x_1x_2 : y_1y_2 : z_1z_2 = \Theta_1 A - \Delta : \Theta_1 B - \Delta : \Theta_1 C - \Delta$, also $x_2 : y_2 : z_2 = A\sin A + H\sin B + G\sin C : H\sin A + B\sin B + F\sin C : G\sin A + F\sin B + C\sin C$, (SALMON'S *Conics*, Art. 293), (x_2, y_2, z_2) being the coordinates of the infinitely distant focus.

The equation of the axis is

$$x(y_1z_2 - y_2z_1) + y(z_1x_2 - z_2x_1) + z(x_1y_2 - x_2y_1) = 0,$$

where $x_1 : y_1 : z_1 = \dfrac{\Theta_1 A - \Delta}{A\sin A + H\sin B + G\sin C} : \dfrac{\Theta_1 B - \Delta}{H\sin A + B\sin B + F\sin C}$

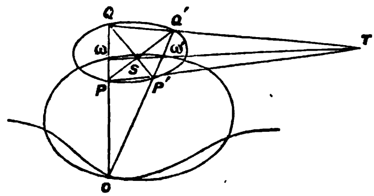
$$: \frac{\Theta_1 C - \Delta}{G\sin A + F\sin B + C\sin C};$$

and $x_2 : y_2 : z_2$ are as given above.

I may mention that $\Theta_1\Sigma - \Delta\Omega = 0$ is in general the tangential equation of the envelope of a chord such that the tangents to Σ at its extremities cut at right angles.

A property of a Cubic and the Polar Conic of a point upon it. By Thomas Cotterill, M.A.

Theorem.—*If a transversal through a point on a cubic meets the curve again in two points, and the polar conic of the point to the cubic in a third point; then the tangents to the curves at these three points are concurrent.*



For, let lines OPQ, OP′Q′ through O a point on the cubic meet it again in PQ, P′Q′, and the polar conic of O in ω, ω' respectively, so that in the limit PP′, QQ′, $\omega\omega'$ respectively coincide; and let PQ′, QP′ intersect in S, and PP′, QQ′ in T. Then by the harmonic properties of the quadrilinear of which PQ, P′Q′, ST are opposite points, Oω, Oω' are harmonic conjugates to PQ and P′Q′; hence, by the harmonic properties of the polar conic, $\omega\omega'$ must be points upon it. Now, in the limit, when PP′, QQ′ and $\omega\omega'$ respectively coincide, the lines through them become tangents to the curves, which therefore meet in a point.

One of the consequences of this simple property is, that a point on the conic and the corresponding intersection of the tangents are inverse points to a self-conjugate triangle of the conic, to which the cubic is its own inverse: so that if $ax^2+by^2+cz^2=0$ is the equation to the conic referred to this triangle, the locus of the intersection of the tangents is of the form

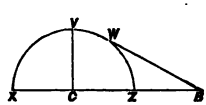
$$\alpha y^2z^2+\beta z^2x^2+\gamma x^2y^2=0.$$

1793. (Proposed by M. W. Crofton, B.A.)—1. If two sides of a triangle be given, and the third side be taken at random (from among all its possible values), find the probability that the triangle is acute-angled.

2. Two points are taken at random in a given line (l); find the probability of the distance between them exceeding a given length (c).

3. In a plane triangle if arbitrary values be taken for a, b, and C (two sides and the contained angle), and the extreme limit be the same for a and b; find the probability that the triangle is obtuse-angled.

Solution by the Proposer; E. Fitzgerald; *and others.*



1. Given a, b, two sides of a triangle, the third being taken arbitrarily; to find the chance of the triangle being acute.

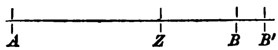
Let CB $= a$, the greater of the two given sides; from C as centre draw a circle of radius b; the vertex is on the semi-circle XVZ. Now, the triangle is acute, only if the vertex is on the arc VW, BW being a tangent. Hence, [since the *possible*

values of c range over BX$-$BZ, and those which give the favourable cases over BV$-$BW,] it is easily seen that the required chance is

$$p = \frac{\sqrt{(a^2+b^2)} - \sqrt{(a^2-b^2)}}{2b}.$$

If $a = \sqrt{5}$ and $b = 2$, we have $p = \frac{1}{2}$, or it is then an *even* chance that the triangle is acute.

2. Let the given line AB take an increment $dl =$ BB$'$, c remaining unchanged; if F be the measure of the favourable cases, make BZ$=c$, and we have $d\text{F} = \text{AZ}\,.\,dl = (l-c)\,dl$,



therefore $\text{F} = \frac{1}{2}l^2 - cl + \text{C}$, C being a function of c.

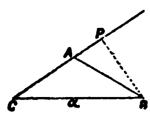
Now, if $l = c$, F $= 0$; hence $\text{C} = \frac{1}{2}c^2$, and $\text{F} = \frac{1}{2}(l-c)^2$.

Hence we have $p = \dfrac{\text{F}}{\frac{1}{2}l^2} = \left(\dfrac{l-c}{l}\right)^2.$

Otherwise: suppose one point chosen at a distance x from one end of the line, and suppose the distance c set off from this towards each end of the line; then any point chosen between either of these points and the ends will give a favourable case with the point x. The chance of one of these is plainly $\dfrac{x-c}{l}$, and of the other $\dfrac{l-x-c}{l}$. That the *first* should give a real chance, x must lie between c and l; and for the *second* x must lie between 0 and $l-c$. Now the chance of x being taken in any particular position is $\dfrac{dx}{l}$; and consequently the sum of the compound chances gives for the chance required

$$p = \int_0^{l-c} \frac{l-x-c}{l}\,.\,\frac{dx}{l} + \int_c^l \frac{x-c}{l}\,.\,\frac{dx}{l} = \left(\frac{l-c}{l}\right)^2.$$

3. Let ABC be the triangle; the chance of C being obtuse is $\frac{1}{2}$. To find the chance for A, let k be the extreme limit of value for a and b, draw BP perpendicular to b; then if a and C are supposed fixed, the chance that A is obtuse is clearly $\dfrac{a \cos \text{C}}{k}$.



Let now C vary from $\frac{1}{2}\pi$ to 0, a remaining fixed; then

$$p = \frac{a}{k}\int_0^{\frac{1}{2}\pi} \cos \text{C}\,.\,\frac{d\text{C}}{\pi} = \frac{a}{\pi k}.$$

If a vary now from k to 0, the chance that A is obtuse is

$$p = \int_0^k \frac{a}{\pi k}\,.\,\frac{da}{k} = \frac{1}{2\pi}.$$

The same value holds for the angle B, hence the whole probability that the triangle is obtuse-angled is $\dfrac{1}{2} + \dfrac{1}{\pi}$.

1970. (Proposed by Professor Cayley.)—Find the conditions in order that the conics
$U = (a, b, c, f, g, h)(x, y, z)^2 = 0,\quad U' = (a', b', c', f', g', h')(x, y, z)^2 = 0,$
may have double contact.

Solution by the Proposer.

The coefficients of the two conics must be so related that for a properly determined value of θ we shall have identically $U - \theta U' = (\lambda x + \mu y + \nu z)^2$; but when this is so, the inverse coefficients of the quadric function $U - \theta U'$ are each $= 0$; that is, writing

$(A, B, C, F, G, H) = (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch)$
$(A', B', C', F', G', H') = (b'c' - f'^2, \ldots, \ldots, g'h' - a'f', \ldots, \ldots)$
$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}) = (bc' + b'c - 2ff', \ldots, gh' + g'h - af' - a'f, \ldots),$

then we have the six equations $A - \theta\mathfrak{A} + \theta^2 A' = 0$, &c.

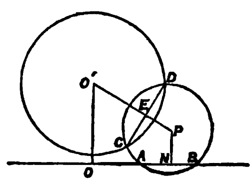
Or, eliminating θ, the required conditions are

$$\left\| \begin{matrix} A, & B, & C, & F, & G, & H \\ A', & B', & C', & F', & G', & H' \\ \mathfrak{A}, & \mathfrak{B}, & \mathfrak{C}, & \mathfrak{F}, & \mathfrak{G}, & \mathfrak{H} \end{matrix} \right\| = 0,$$

equivalent to three relations between the two sets of cofficients.

1972. (Proposed by R. Tucker, M.A.)—Find the envelope and locus of centres of a system of circles which intercept constant lengths on a fixed line and a fixed circle.

Solution by the Proposer.



Let the fixed straight line and the perpendicular on it from the centre (O') of the given circle be the axes of coordinates; and let the variable circle (P) intersect the line in AB ($=2a$) and the circle in CD (chord CD $= 2k$). Then, if $O'O = b$, $O'D = c$, we have

$$a^2 + y^2 = (AP)^2 = k^2 + (PE)^2,$$
$$(O'P)^2 = x^2 + (b-y)^2,\ (O'E)^2 = c^2 - k^2;$$

hence we have for the locus of P (x, y)

$$\sqrt{\{x^2 + (b-y)^2\}} = \sqrt{(c^2 - k^2)} + \sqrt{(a^2 + y^2 - k^2)},$$

or, $$x^2 - 2by + \lambda^2 = \mu\sqrt{(a^2 + y^2 - k^2)},$$

if $$\lambda^2 = 2k^2 + b^2 - c^2 - a^2 \text{ and } \mu = 2\sqrt{(c^2 - k^2)},$$

that is, $$x^4 + 4b^2y^2 + \lambda^4 - 4bx^2y + 2\lambda^2x^2 - 4b^2\lambda y = \mu^2(a^2 + y^2 - k^2),$$

or, $$x^4 + 2x^2(\lambda^2 - 2by) + 4b^2y^2 + \lambda^4 - 4b\lambda^2 y - \mu^2(a^2 + y^2 - k^2) = 0.$$

If $a = k$, this reduces to the parabolas

$$x^2 - 2by + \lambda^2 = \pm\mu y \ldots\ldots\ldots\ldots (i),$$

and if $c = k$ or $\mu = 0$, we have the parabola

$$x^2 - 2by + \lambda^2 = 0 \quad\text{.........................}\quad \text{(ii)}.$$

For the envelope of the circles we have, in the case (i),

$$X^2 + Y^2 - 2Xx - 2Yy + x^2 = a^2, \qquad x^2 - 2by + \lambda^2 = \pm\, \mu y;$$

hence $\dfrac{dy}{dx} = \dfrac{2x}{2b \pm \mu} = \dfrac{x - X}{y}$, and $x = \dfrac{b'X}{b' - 2Y}$, if $b' = 2b \pm \mu$.

Substituting and reducing, we get

$$\left\{ b'(Y^2 - a^2) - 2\lambda^2 Y \right\} (b' - 2Y)^2 = 2b'X^2Y\,(b' - 2Y),$$

or,
$$2b'X^2Y = (b' - 2Y)\left\{ b'Y^2 - 2\lambda^2 Y - a^2 b' \right\},$$

hence the envelope on this supposition is a cubic curve.

1990. (Proposed by Professor SYLVESTER.)—Prove that the locus of one set of foci of all conics passing through four points on a circle is a circular cubic.

Solution by W. S. BURNSIDE, M.A.

Before proceeding with the solution, it may be well to re-state the question in such a form as to give a cue to what will follow.

The locus of the foci of conics passing through four points is in general a curve of the sixth degree; but when the points lie on a circle, the locus resolves itself into two circular cubics, having the four points for foci; also the real foci of the conics lie on one cubic, and the imaginary foci on the other cubic.

MOEBIUS has given the following relation connecting four points, A, B, C, D, on a conic with the focus O:

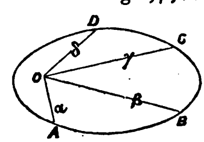
$$OA\,(BCD) - OB\,(CDA) + OC\,(DAB) - OD\,(ABC) = 0 \quad\text{.........}\quad \text{(I.)},$$

which is easily seen to be true, by writing the coordinates of the four points in the equation of the conic $\rho = \lambda x + \mu y + \nu$ (origin at the focus), and eliminating λ, μ, ν; and this is the form under which we will consider the equation of the locus of the foci.

FEUERBACH has given the relation connecting four points on a circle with any arbitrary point, viz.

$$OA^2\,(BCD) - OB^2\,(CDA) + OC^2\,(DAB) - OD^2\,(ABC) = 0 \quad\text{......}\quad \text{(II.)},$$

which also follows by substituting the coordinates of the four points in the equation of the circle $\rho^2 = \lambda x + \mu y + \nu$ (origin at O), and eliminating λ, μ, ν.



We now write the last two relations in the shorter forms

$$l\alpha + m\beta + n\gamma + r\delta = 0 \quad\text{..........}\quad (1),$$

$$l\alpha^2 + m\beta^2 + n\gamma^2 + r\delta^2 = 0 \quad\text{..........}\quad (2),$$

with
$$l + m + n + r \equiv 0 \quad\text{..........}\quad (3).$$

Eliminating δ, we have

$$r\,(l\alpha^2 + m\beta^2 + n\gamma^2) + (l\alpha + m\beta + n\gamma)^2 = 0 \quad\text{................}\quad (4).$$

Forming now the discriminant of the left-hand side of the equation, it is found to be $lmnr^2(l+m+n+r)$, which vanishes in virtue of (3); whence the first part of the theorem is true, as (4) is resolvable into

$$(l_1\alpha+m_1\beta+n_1\gamma)\,(l_2\alpha+m_2\beta+n_2\gamma)=0.$$

Again, these factors are actually

$$lm\,(\alpha-\beta)+ln\,(\alpha-\gamma)\pm(lmnr)^{\frac{1}{2}}\,(\beta-\gamma),$$

showing that $l_1+m_1+n_1\equiv 0$, and $l_2+m_2+n_2\equiv 0$, or that the two constituents of the locus are circular cubics having the points A, B, C for foci.

From $l_1\alpha+m_1\beta+n_1\gamma=0$ combined with $l_1+m_1+n_1\equiv 0$, we see that, if the point (α, β, γ) lies on this cubic, so does the point $(\alpha+h, \beta+h, \gamma+h)$; and these points may be considered as conjugate foci.

As we might equally well have retained the vectors (β, γ, δ) only in the equations of the cubics, it appears that all four points are foci of each of them.

2230. (Proposed by M. Hermite.)—Soit $F(x)$ un polynome qui reste positif pour toutes les valeurs réelles de la variable; il en sera de même du polynome suivant:

$$\Phi(x)=F(x)+aF'(x)+a^2F''(x)+a^3F'''(x)+\&c.$$

quel que soit la constante a.

Et si le polynome $F(x)$ est quelconque, la plus grande racine de l'équation $\Phi(x)=0$ sera inférieure à la plus grande des racines de $F(x)=0$, si la constante a est positive.

Plus généralement, soit $\Theta(x)=1+\alpha x+\beta x^2+\&c.=0$ une equation dont toutes les racines *sont réelles et positives;* si l'on fait

$$\frac{1}{\Theta(x)}=1+ax+bx^2+cx^3+\&c.,$$

la plus grande racine réelle de

$$\Phi(x)=F(x)+a\,F'(x)+b\,F''(x)+c\,F'''(x)+\&c.$$

sera audessous de la plus grande racine réelle de $F(x)=0$, et si le polynome $F(x)$ est positif quel que soit x, il en sera de même de $\Theta(x)$. Seulement alors il suffit que toutes les racines de $\Theta(x)=0$ soient réelles, sans être toutes positives.

Solution by Thomas Savage, M.A.

1. It is evident that $\Phi(x)-a\dfrac{d}{dx}\Phi(x)=F(x)$, or multiplying both sides by $\dfrac{1}{a}\epsilon^{-\frac{x}{a}}$, we have $\dfrac{d}{dx}\left\{-\epsilon^{-\frac{x}{a}}\Phi(x)\right\}=\dfrac{1}{a}\epsilon^{-\frac{x}{a}}F(x)$.

Now if a be positive, the right-hand side of this equation is always positive. Hence the function $-\epsilon^{-\frac{x}{a}}\Phi(x)$ must constantly increase as x increases from $-\infty$ to $+\infty$, and therefore if it is negative when $x=+\infty$ it is always

negative. But it is negative when $x = +\infty$; for the terms which involve the highest powers of x in $F(x)$ and $\Phi(x)$ are the same, and when x is very large the sign of these terms determines the signs of $F(x)$ and $\Phi(x)$; hence when x is large $\Phi(x)$ is positive, and $-\epsilon^{-\frac{x}{a}}\Phi(x)$ is negative. And since $-\epsilon^{-\frac{x}{a}}\Phi(x)$ is always negative, $\Phi(x)$ is always positive.

Again, let a be negative. It is easily seen from (1) that the function $-\epsilon^{-\frac{x}{a}}\Phi(x)$ continually diminishes as x increases from $-\infty$ to $+\infty$, and as before, when $x = -\infty$, $F(x)$ and $\Phi(x)$ have the same sign, or $-\epsilon^{-\frac{x}{a}}\Phi(x)$ is negative; it is therefore always negative and $\Phi(x)$ is therefore always positive.

2. Suppose now that $F(x)$ is not always positive, but that its greatest root is k. Then for values of x greater than k, $F(x)$ is always positive; and if a be positive, it may be shown as before, that as x increases from k to $+\infty$, the function $-\epsilon^{-\frac{x}{a}}\Phi(x)$ also continually increases; and when x is very large, this function is negative; it is therefore negative for all values of x greater than k, or $\Phi(x)$ is positive for all such values, and the greatest root of $\Phi(x)$ is less than k.

It is also easily seen that if a be negative, *and the degree of* $F(x)$ *be even*, the numerically greatest negative root of $\Phi(x)$ is (numerically) less than the greatest negative root of $F(x)$.

3. Again, let $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\ldots$ be the roots of the equation $\Theta(x) = 0$. Then we have (separating the symbols of operation)

$$\Phi(x) = \frac{F(x)}{\Theta\left(\frac{d}{dx}\right)} = \frac{F(x)}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)\left(1-r\frac{d}{dx}\right)\ldots},$$

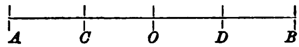
and if $p, q, r\ldots$ be all positive, and k the greatest root of $F(x)$, it has been shown that the greatest root of $\frac{F(x)}{1-p\frac{d}{dx}}$ is less than k. Let it be equal to k'.

Then the greatest root of $\frac{F(x)}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)}$ is less than k'. Proceeding in this way we see that the greatest root of $\Phi(x)$ is less than k.

If $F(x)$ be always positive, then, whether p, q, r, &c., be positive or negative, $\frac{F(x)}{1-p\frac{d}{dx}}$ will be always positive; therefore also $\frac{F(x)}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)}$ will be always positive; and thus it may be proved that $\frac{F(x)}{\Theta\left(\frac{d}{dx}\right)}$, or $\Phi(x)$, is always positive.

2257. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—A line, fixed in length and position, is cut at two variable points into three segments the sum of whose squares is constant; required the locus of the vertex of the equilateral triangle described on the middle segment as base.

Solution by S. WATSON; W. H. LAVERTY; W. CHADWICK; E. MCCORMICK; *and others.*



Let O be the middle point of the given line AB, and C, D the dividing points. Take OB and a perpendicular to it at O for axes, and put AB $= a$, AC $= z$, CD $= z'$; then

$$z^2 + z'^2 + (a - z - z')^2 = c^2 \text{ (a constant)} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Also the coordinates of the vertex of the equilateral triangle on CD are

$$x = z + \tfrac{1}{2}z' - \tfrac{1}{2}a, \text{ and } y = \tfrac{1}{2}z'\sqrt{3} \quad \ldots\ldots\ldots\ldots\ldots (2);$$

hence eliminating z, z' from (1) and (2), the result, viz.

$$x^2 + y^2 - \tfrac{1}{3}\sqrt{3}\,.\,ay = \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2,$$

is the equation of the required locus, and represents a circle.

[When $c = a$, the circle passes through A and B; when $c^2 = \frac{1}{2}a^2$ the circle touches AB; when $c^2 = \frac{1}{3}a^2$, the circle becomes a point; and when $c^2 < \frac{1}{3}a^2$, the circle becomes imaginary, showing that the line cannot be divided so that the squares on the three segments shall be less than one-third of the square on the whole line, as is indeed obvious from other considerations.]

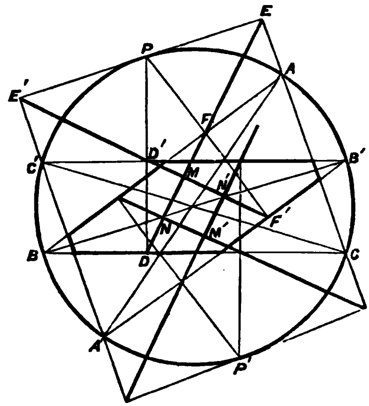
1495. (Proposed by HUGH GODFRAY, M.A.)—Show that $\frac{1}{6}n(n-1)(n-2)$ points can always be so arranged in a plane that they shall be situated by eights in $\frac{1}{24}n(n-1)(n-2)(n-3)$ circles.

Solution by the PROPOSER.

Let us consider n points in space. Through any four of these a sphere may be drawn, and the n points taken in sets of four will give $\frac{1}{24}n(n-1)(n-2)(n-3)$ spheres. Again, through any three of the points a circle may be drawn, and the n points taken three at a time will furnish $\frac{1}{6}n(n-1)(n-2)$ circles; and it is easily seen that each sphere will have four of the circles on its surface. Now, draw a plane so as to cut all the spheres and all the circles. The intersection with each circle will give *two* points on the plane; and the intersection with each sphere will give a circle on the plane, which circle will contain the *eight* points of intersection with the four circles on that sphere. Therefore there will be $\frac{1}{3}n(n-1)(n-2)$ points, situated by *eights* on $\frac{1}{24}n(n-1)(n-2)(n-3)$ circles.

1939. (Proposed by R. Tucker, M.A.)—Prove that the two feet-perpendicular lines corresponding to any point on the circumscribing circle of a pair of *diametral* triangles intersect at right angles on an ellipse tangential to the six sides. [ABC, A′B′C′ are called *diametral* triangles when AA′, BB′, CC′ intersect in the centre of the common circumscribing circle.]

Solution by James Dale; *and others.*



Drawing the figure, let DEF, D′E′F′, be the feet-perpendiculars of the triangles ABC, A′B′C′, corresponding to the point P; and let 2θ be the angle subtended by AP at the centre of the circle. Then in the triangle D′DM, the angle D′DM = FBP = θ; the angle DD′M = PD′E′ = PC′E′ = PB′A′ = $90^\circ - \theta$; therefore DD′M + D′DM = 90°, and consequently the angle at M is a right angle.

Taking ABC as the triangle of reference, the trilinear coordinates of M are

$$x = \text{DM}\cos\theta = \text{DD}'\cos^2\theta,$$
$$y = \text{EM}\cos\text{MEE}' = \text{EM}\cos(\text{C}-\theta) = \text{EE}'\cos^2(\text{C}-\theta),$$
$$z = \text{FM}\cos\text{PFE} = \text{FM}\cos(\text{B}+\theta) = \text{FF}'\cos^2(\text{B}+\theta);$$

also DD′ = 2R cos A, EE′ = 2R cos B, FF′ = 2R cos C;
therefore $a(x\sec \text{A})^{\frac{1}{2}} + b(y\sec\text{B})^{\frac{1}{2}} + c(z\sec\text{C})^{\frac{1}{2}}$

$$= (2\text{R})^{\frac{1}{2}}\left\{\pm a\cos\theta \pm b\cos(\text{C}-\theta) \pm c\cos(\text{B}+\theta)\right\} = 0.$$

The equation $a(x\sec\text{A})^{\frac{1}{2}} + b(y\sec\text{B})^{\frac{1}{2}} + c(z\sec\text{C})^{\frac{1}{2}} = 0$ represents an ellipse having its centre at the centre of the circumscribing circle, and touching the sides BC, CA, AB, B′C′, C′A′, A′B′ at the points where these lines are intersected by perpendiculars from A′, B′, C′, A, B, C.

If we draw the feet-perpendiculars corresponding to the *diametral* point P′, these, with the two corresponding to P, form a rectangle MNM′N′, whose angles M, M′ lie on the tangential ellipse, while N, N′ lie respectively on the nine-point circles of the triangles ABC, A′B′C′.

2226. (Proposed by H. Tomlinson.)—If a conic have double contact with two other conics, prove that the chords of intersection of these two conics both pass through the intersection of the chords of contact of the two conics with the first conic.

I. *Solution by* A. Cohen, B.A.; E. McCormick; H. Tomlinson; W. Chadwick; *the* Proposer; *and others.*

Let $S = 0$ be the equation to the conic. Then the equations to the conics having double contact with the conic $S = 0$ will be of the form $S - k^2\alpha^2 = 0$, $S - k'^2\alpha'^2 = 0$, and by subtracting we find that the chords of intersection of these two conics have for their equations $k\alpha - k'\alpha' = 0$, $k\alpha + k\alpha' = 0$, which therefore both pass through the intersection of $\alpha = 0$, $\alpha' = 0$, or of the chords of contact of the two conics with the first.

II. *Solution by* Archer Stanley.

Let the conic (S) have double contact in A and B with (Σ), and let the latter be touched by the third conic (S′) in A′ and B′. The intersection p of AB and A′B′ will *obviously* have the same polar P relative to all three conics. Hence if a be an intersection of S and S′, and pa intersect P in π, it will intersect both S and S′ again in the harmonic conjugate of a relative to p and π; that is to say, $p\pi$ will be a common chord of S and S′, and in like manner the remaining common chord passes through p.

1521. (Proposed by J. M. Wilson, M.A., F.G.S.)—Show that, in a geometric progression of an odd number of terms, the arithmetic mean of the odd-numbered terms is greater than the arithmetic mean of the even-numbered terms, if the common ratio be any positive rational quantity not equal to unity.

Solution by C. Taylor, M.A.

Although Dr. Ingleby's solution (*Reprint*, Vol. V., p. 26), leaves nothing to be desired in point of elegance, another inductive proof may perhaps interest those to whom the theorem has "presented great difficulties."

Let F_n denote the fraction $\dfrac{1 + a^2 + a^4 + \dots + a^{2n}}{a + a^3 + \dots + a^{2n-1}}$, and D_n its denominator;

then we have to prove that $F_n \equiv \dfrac{1 + aD_n}{D_n} > \dfrac{n+1}{n}$, when $a >< 1$.

Since the *arithmetical mean of any number of positive quantities is greater than the geometrical mean,* (see Todhunter's *Algebra*, Art. 676,)

therefore $$D_n > na^n \text{ and } D_{n+1} > (n+1)\,a^{n+1}.$$

But $$F_n - F_{n+1} = \frac{1}{D_n} - \frac{1}{D_{n+1}} = \frac{a^{2n+1}}{D_n \cdot D_{n+1}} < \frac{1}{n\,(n+1)},$$

therefore $$F_{n+1} > F_n - \frac{1}{n\,(n+1)}, \text{ and, } a\ fortiori,$$

if $$F_n > \frac{n+1}{n}, \text{ we have } F_{n+1} > \frac{n+1}{n} - \frac{1}{n\,(n+1)} > \frac{n+2}{n+1}.$$

Hence inductively, we have, $F_n > \frac{n+1}{n}$.

It may be proved that

$$\frac{n}{n+1} F_n > \frac{n-1}{n} F_{n-1} > \ldots\ldots > \frac{1}{2}\left(a + \frac{1}{a}\right).$$

2222. (Proposed by S. BILLS.)—Prove that if $2x+1$ be any *prime* number, and if a and b be any two unequal whole numbers, not multiples of $2x+1$, then either $a^x + b^x$ or $a^x - b^x$ will be exactly divisible by $2x+1$.

Solution by the REV. J. BLISSARD; T. POOLEY; W. CHADWICK; J. DALE; E. MCCORMICK; *the* PROPOSER; *and others.*

By FERMAT'S Theorem, $\frac{a^{2x}-1}{2x+1}$ and $\frac{b^{2x}-1}{2x+1}$ are integral. Hence one of the expressions $\frac{a^x-1}{2x+1}$, $\frac{a^x+1}{2x+1}$ must be integral, as also one of the expressions $\frac{b^x-1}{2x+1}$, $\frac{b^x+1}{2x+1}$; and whichever of the first pair be integral and whichever of the second pair, by adding or subtracting as the case may require, either $\frac{a^x-b^x}{2x+1}$, or $\frac{a^x+b^x}{2x+1}$ must be integral. Thus, if $\frac{a^x-1}{2x+1}$ and $\frac{b^x+1}{2x+1}$ are integers, then by adding, $\frac{a^x+b^x}{2x+1}$ must be an integer. For example, if $x = 8$, then either a^8+b^8 or a^8-b^8 is exactly divisible by 17, &c.

1904. (Proposed by J. GRIFFITHS, M.A.)—Let G′ denote the inverse of the centre of gravity G of any triangle ABC; H the equilateral hyperbola which passes through the points A, B, C, G; E the ellipse which touches the sides of the triangle at the points where they are intersected by the lines AG′, BG′, CG′; show that the nine-point circle of the triangle touches the common tangents of the two curves H and E.

I. *Solution by* W. H. LAVERTY.

Let ABC be the triangle of reference. Now, we know that if

$$\Sigma Ax^2 + 2\Sigma Dyz = (A, B, C, D, E, F)\,(x, y, z)^2 = 0$$

be the equation to a conic, and if A′, B′, C′, D′, E′, F′ are the coefficients of

A, B, C, D, E, F found by expanding $\begin{vmatrix} \mathrm{AFE} \\ \mathrm{FBD} \\ \mathrm{EDC} \end{vmatrix}$; we have $(A', B', C', D', E', F')(l, m, n)^2 = 0$ for the condition that $lx+my+nz=0$ may touch the conic.

Now we have for G and G′ respectively

$$\frac{x}{a^{-1}} = \frac{y}{b^{-1}} = \frac{z}{c^{-1}}; \text{ and } \frac{x}{a} = \frac{y}{b} = \frac{z}{c};$$

and the equations to H and E are respectively

$yz \sin(B-C) + zx \sin(C-A) + xy \sin(A-B) = 0$, and $\Sigma \frac{x^2}{a^2} - 2\Sigma \frac{yz}{bc} = 0$;

and therefore the conditions that $lx+my+nz=0$ should be a tangent to H and E are

$$\Sigma\left(\frac{(b^2-c^2)^2}{a^2} l^2\right) - 2\Sigma\left(\frac{(c^2-a^2)(a^2-b^2)}{bc} mn\right) = 0,$$

and $$bc \cdot mn + ca \cdot nl + ab \cdot lm = 0.$$

Dividing the first of these by 4 and subtracting it from the other, we have

$$\Sigma \frac{(b^2-c^2)^2}{4a^2} l^2 - \Sigma\left(\frac{bc}{2} + a^2 \cdot \cos A\right) mn = 0,$$

which will be found to be the condition that $lx+my+nz=0$ should touch the nine-point circle $\Sigma a \cos A \cdot x^2 - \Sigma a \cdot yz = 0$.

II. *Solution by* JAMES DALE.

The equations in trilinear coordinates of the hyperbola, ellipse, and circle are as follows:—

$$\frac{b^2-c^2}{a} yz + \frac{c^2-a^2}{b} zx + \frac{a^2-b^2}{c} xy = 0 \quad \ldots\ldots\ldots\ldots\ldots (\mathrm{H}),$$

$$\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(\frac{y}{b}\right)} + \sqrt{\left(\frac{z}{c}\right)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{E}),$$

$$a \cos A \cdot x^2 + b \cos B \cdot y^2 + c \cos C \cdot z^2 - (ayz + bzx + cxy) = 0 \ldots\ldots (\mathrm{C}).$$

The conditions that the line $lx+my+nz=0$ should touch (H), (E), (C) are, respectively,

$$\frac{(b^2-c^2)^2}{a^2} l^2 + \frac{(c^2-a^2)^2}{b^2} m^2 + \frac{(a^2-b^2)^2}{c^2} n^2$$
$$-2(c^2-a^2)(a^2-b^2)\frac{mn}{bc} - 2(a^2-b^2)(b^2-c^2)\frac{nl}{ca} - 2(b^2-c^2)(a^2-b^2)\frac{lm}{ab} = 0,$$
$$\ldots\ldots\ldots\ldots (1),$$

$$bc \cdot mn + ca \cdot nl + ab \cdot lm = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(b^2-c^2)\frac{l^2}{a^2} + (c^2-a^2)\frac{m^2}{b^2} + (a^2-b^2)\frac{n^2}{c^2} - 2(b^2c^2+c^2a^2+a^2b^2-a^4)\frac{mn}{bc}$$
$$-2(b^2c^2+c^2a^2+a^2b^2-b^4)\frac{nl}{ca} - 2(b^2c^2+c^2a^2+a^2b^2-c^4)\frac{lm}{ab} = 0 \quad \ldots\ldots (3).$$

Now $(1)+4(2) \equiv (3)$; hence the values of l, m, n, which satisfy (1) and (2) simultaneously, also satisfy (3); in other words, (1), (2), (3) are inscribed in the same quadrilateral.

1839. (Proposed by W. S. BURNSIDE, M.A.)—If normals be drawn to a conic at the points P, Q; show that a parabola can be described, touching these two normals, the chord PQ, and the axes of the conic; the diameter conjugate to the chord being the directrix. Also verify the following determination of the common tangents to these curves; through the pole of the chord draw the four normals to the conic, the tangents at their feet are the common tangents required.

I. *Solution by* JAMES DALE.

Let $Ax^2+By^2=1$ be the equation to a central conic, having the principal axes for axes of coordinates; and (x', y'), (x'', y'') the coordinates of any two points P, Q on the curve. Then the directrix of the parabola touching the normals at P and Q, the chord PQ, and the major axis, is the line joining the intersection of the perpendiculars of the triangles formed by the normals and the chord, and by the normals and the major axis.

The perpendiculars from P, Q on the normals at Q, P respectively are

$$y-y' = -\frac{A}{B}\left(\frac{x''}{y''}\right)(x-x'), \quad y-y'' = -\frac{A}{B}\cdot\frac{x'}{y'}(x-x'') \ldots\ldots (1, 2).$$

The intersection of the perpendiculars of the triangle formed by the normals and the major axis is given by the equations

$$y = -\frac{A}{B}\cdot\frac{x''}{y''}\left(x-\frac{B-A}{A}x'\right), \quad y = -\frac{A}{B}\cdot\frac{x'}{y'}\left(x-\frac{B-A}{A}x''\right) \ldots (3, 4);$$

also, y'' (1)$-y'$ (2), or y'' (3)$-y'$ (4), gives for the equation of the directrix

$$y = -\frac{A}{B}\left(\frac{x'-x''}{y'-y''}\right)x, \quad \text{or } y = \frac{y'+y''}{y'+x''}x \ldots\ldots\ldots\ldots (5, 6).$$

Now, (6) passes through the centre and the middle of the chord, and is therefore the conjugate of the chord; and as the minor axis cuts the major axis at right angles, it follows that the minor axis is also a tangent to the same parabola.

For the second part; let (x''', y''') be any point on the conic, then the condition that the intersection of the perpendiculars of the triangle formed by the tangent at (x''', y''') and the normals at (x', y'), (x'', y'') should lie upon the directrix is readily found to be

$$\frac{A}{B}x'''(x'-x'') + \frac{B}{A}y'''(y'-y'') = (A-B)x'''y'''(x'y''-y'x'');$$

and this is also the condition that the point (x''', y''') should lie upon the hyperbola, which, by its intersection with the conic, determines the feet of the normals drawn through the pole of the chord PQ.

II. *Solution by the* PROPOSER.

1. Let the conic be referred to its axes, and the point (α, β) be the pole of the chord. Now, if $\lambda x+\mu y+\nu=0$ be the equation of the normal at one of the points whose tangent passes through the point (α, β), we easily find that $$C^2\lambda\mu+\alpha\mu\nu-\beta\nu\lambda=0,$$
by comparing with the known equation of the normal.

2. Reciprocally, if $\lambda x+\mu y+\nu=0$ be the equation of the tangent at one of the points whose normal passes through the point (α, β), we find the same tangential equation

$$C^2\lambda\mu+\alpha\mu\nu-\beta\nu\lambda=0.$$

3. This equation is satisfied by $(\lambda=0, \mu=0)$, $(\mu=0, \nu=0)$, $(\lambda=0, \nu=0)$; hence the curve of the second degree represented by this equation touches the line at infinity and the axes of the conic. It is also satisfied by $\lambda=\frac{\alpha}{a^2}$, $\mu=\frac{\beta}{b^2}$, $\nu=-1$; hence this curve touches the chord PQ. So we have proved that a parabola can be described touching the lines described in the question.

4. It remains to show that (α, β) is a point in the directrix, or that the tangents drawn therefrom to the parabola are perpendicular. If $\lambda x+\mu y+\nu=0$ be the equation of one of the tangents, then $\lambda\alpha+\mu\beta+\nu=0$; hence, eliminating ν from the equation $C^2\lambda\mu+\alpha\mu\nu-\beta\nu\lambda=0$, we see that the sum of the coefficients of λ^2 and μ^2 is zero, or the tangents to the parabola from the point (α, β) are at right angles: also the centre is plainly another point on the directrix, since it touches the axes: hence the directrix is determined.

5. Denoting the tangential equations of the conic, the two circular points at infinity, and the pole of the chord, by Σ, Ω, Π, respectively, the equation of the parabola may be written as the Jacobian of these functions, which becomes in the general case [$\alpha x+\beta y+\gamma z=0$ equation of chord, $(a, b, c, f, g, h)(x, y, z)^2=0$ equation of conic] when Δ is divided off,

$$\begin{vmatrix} a & h & g \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{vmatrix}\Omega_1+\begin{vmatrix} h & b & f \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{vmatrix}\Omega_2+\begin{vmatrix} g & f & c \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{vmatrix}\Omega_3=0, \quad \left(\Omega_1=\frac{d\Omega}{d\lambda}, \text{ \&c.}\right).$$

In conclusion, I should state that the above results may be easily proved by reciprocation.

1809. (Proposed by G. DARBOUX.)—On circonscrit à un triangle quelconque une courbe du second degré telle que les normales aux trois sommets du triangle passent par un point. On demande de prouver que le lieu de ce point est une courbe à centre du troisième ordre. Déterminer cette courbe.

Solution by J. DALE; *and* A. COHEN, B.A.

Let $lyz+mzx+nxy=0$ be the equation of any circumscribing conic, (x', y', z') the coordinates of the point of intersection of the normals.

Then the tangents at A, B, C are respectively

$$\frac{y}{m}+\frac{z}{n}=0, \frac{z}{n}+\frac{x}{l}=0, \frac{x}{l}+\frac{y}{m}=0 \ldots\ldots(1, 2, 3);$$

and the normals at the same points are

$$\frac{y}{y'}-\frac{z}{z'}=0, \quad \frac{z}{z'}-\frac{x}{x'}=0, \quad \frac{x}{x'}-\frac{y}{y'}=0 \ldots(4, 5, 6).$$

But since (1, 2, 3) are respectively perpendicular to (4, 5, 6), we have

$$m(y'+z'\cos A)=n(y'\cos A+z'),\quad n(z'+x'\cos B)=l(z'\cos B+x'),$$
$$l(x'+y'\cos C)=m(x'\cos C+y');$$

hence eliminating l, m, n, we get

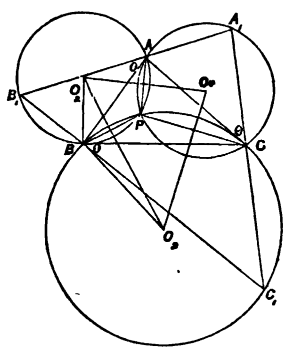
$$(y+z\cos A)(z+x\cos B)(x+y\cos C)=(y\cos A+z)(z\cos B+x)(x\cos C+y),$$

which is the central cubic discussed in the Solution to Question 1958. (*Reprint*, Vol. VI., p. 59.)

1981. (Proposed by T. T. Wilkinson, F.R.A.S.)—If from the angular points of any triangle ABC, lines be drawn making the same constant angles with the adjacent sides, *four* triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, will be formed, which possess the following properties.

(1) The above triangles are all similar to each other and to the triangle ABC; (2) if circles be described about A_1CA, B_1AB, C_1BC, they will meet in a point P; (3) circles described about A_3BA, B_3CB, C_3AC, will meet in another point P_1; (4) if O_1, O_2, O_3 be the centres of the circles in (2) and O_4, O_5, O_6 the centres of those in (3), then the triangles $O_1O_2O_3$, $O_4O_5O_6$, ABC are similar, and the triangle $O_1O_2O_3$ is equal to $O_4O_5O_6$.

Solution by J. Dale; H. Tomlinson; W. H. Laverty; *and others.*



Let B_1C_1, C_1A_1, A_1B_1 make an angle θ towards the same side of BC, CA, AB respectively; then $\angle A_1 = B_1AC - ACA_1 = A$; similarly, $\angle B_1 = B$, and $\angle C_1 = C$; therefore the triangle $A_1B_1C_1$ is similar to ABC. The triangle $A_2B_2C_2$ is formed by making the angle θ towards the *other* side of each of the lines BC, CA, AB; and $A_3B_3C_3$, $A_4B_4C_4$ are formed when B_3C_3, C_3A_3, A_3B_3, and B_4C_4, C_4A_4, A_4B_4 make the same angle θ with CB, AC, BA respectively. The proof that the triangles $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$ are similar to ABC is the same as for $A_1B_1C_1$; hence the theorem (1) is proved.

Let circles be drawn round the triangles A_1CA, B_1AB, C_1BC; and let P be the intersection of A_1CA, B_1AB; then, joining AP, BP, CP, the angles CPA, APB, are the supplements of A and B, consequently BPC is the supplement of C or C_1, and therefore P lies on the circle C_1BC, which proves the theorem (2). The proof of (3) is precisely similar.

[The circles A_2CA, B_2AB, C_2BC are identical with A_1CA, B_1AB, C_1BC; and the circles A_4BA, B_4CB, C_4AC are identical with A_3BA, B_3CB, C_3AC.]

Join BO_2, BO_3, and let R = radius of circle ABC; then we have

$$BO_2 = R\frac{c}{b},\quad BO_3 = R\frac{a}{c},\quad \text{and } \angle O_2BO_3 = A+B,$$

$$\therefore\ (O_2O_3)^2 = R^2\left(\frac{a^2}{b^2} + \frac{c}{b^2} + \frac{2a}{b}\cos C\right) = \frac{R^2}{b^2c^2}(b^2c^2 + c^2a^2 + a^2b^2);$$

$$\therefore\ \frac{O_2O_3}{a} = \frac{R}{abc}(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{2}} = \frac{O_3O_1}{b} = \frac{O_1O_2}{c}, \text{ by symmetry.}$$

Similarly if O_4, O_5, O_6 be the centres of A_3BA, B_3CB, C_3AC, we have

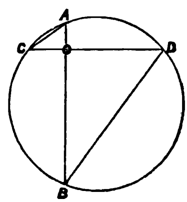
$$CO_5 = R\frac{a}{b},\quad CO_6 = R\frac{b}{c},\ \text{and}\ \angle O_5CO_6 = A + C,$$

$$\therefore\ \frac{O_5O_6}{a} = \frac{R}{abc}(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{2}} = \frac{O_6O_4}{b} = \frac{O_4O_5}{c};$$

therefore the triangles $O_1O_2O_3$, $O_4O_5O_6$, ABC are similar, and since $O_2O_3 = O_5O_6$, &c., the triangle $O_1O_2O_3$ is equal to $O_4O_5O_6$.

2239. (Proposed by C. M. Ingleby, LL.D.)—If a draughtman lie on one of the intersections of the board, show that the sum of the arcs bounding the white sectors is always equal to the sum of the arcs bounding the black sectors.

Solution by W. Chadwick; J. Dale; W. H. Laverty; H. Tomlinson; *the* Proposer; *and others.*



Let the circle represent the draughtsman; O an intersection; AD, BC the arcs bounding the white sectors: then, since the angles BAC, DCA are together equal to a right angle, it follows that the sum of the arcs AD, BC is equal to a semicircle, and therefore equal to the sum of the arcs AC, BD.

Proof of Three Theorems. By Professor Everett.

The following proof of three well known theorems is submitted as preferable in some respects, especially in freedom from tentative processes, to those usually given.

Theorem 1.—$f(a+h) - f(a)$ can be developed in a series of positive integral powers of h, provided that neither $f(x)$ nor any of the derived functions $f'(x), f''(x)$, &c. become infinite or discontinuous between the limits $x = a$ and $x = a + h$.

Theorem 2.—The form of the development is

$$hf'(a) + \frac{h^2}{\underline{|2}}f''(a) + \frac{h^3}{\underline{|3}}f'''(a) + \dots$$

THEOREM 3.—The error committed in neglecting all terms of this series after $\frac{h^n}{\underline{|n}} f^n(a)$ may be represented by $\frac{h^{n+1}}{\underline{|n+1}} f^{n+1}(a+\theta h)$, where θ denotes a positive proper fraction.

We assume, as capable of being independently proved, the three following Lemmas:—

Lemma 1.—If y and z be any variables, the value of $\Sigma(yz)$, taken between such limits that z does not change sign, is equal to $\bar{y}\,\Sigma(z)$, where $\bar{y}$ denotes some quantity intermediate between the algebraically greatest and least values of y.

Lemma 2.—The value of $f(a+h)-f(a)$ is equal to the limit of $\Sigma f'(z)\,\delta z$ from $z=a$ to $z=a+h$, provided that $f'z$ does not become infinite or discontinuous between these limits.

Lemma 3.—The limit of $\Sigma x^n \delta x$ from $x=0$ to $x=h$, when n is a positive integer, is $\frac{h^{n+1}}{\underline{|n+1}}$.

We are now prepared for the following proof, in which, whenever the sign Σ occurs, the summation indicated is to be taken between the limits $x=0$ and $x=h$, and θ always denotes a positive proper fraction but not always the same fraction.

$f(a+h)-f(a) = \text{limit of } \Sigma f'(a+x)\,\delta x = f'(a+\theta h)\,\Sigma\,\delta x = f'(a+\theta h)\,h.$

Hence, writing x for h and f' for f,

$$f'(a+x) = f'(a) + f''(a+\theta x)\,x.$$

Therefore $f(a+h)-f(a)$ being equal to the limit of $\Sigma f'(a+x)\,\delta x$ is equal to the limit of $f'(a)\,\Sigma\,\delta x + f''(a+\theta h)\,\Sigma x\,\delta x$

$$= f'(a)\,h + f''(a+\theta h)\,\frac{h^2}{2}.$$

Hence again, writing x for h and f' for f,

$$f'(a+x) = f'(a) + f''(a)\,x + f'''(a+\theta x)\,\frac{x^2}{2}.$$

Therefore, $f(a+h)-f(a)$ being equal to the limit of $\Sigma f'(a+x)\,\delta x$, is equal to the limit of

$$f'(a)\,\Sigma\,\delta x + f''(a)\,\Sigma x\,\delta x + f'''(a+\theta h)\,\Sigma\,\frac{x^2\,\delta x}{2}$$

$$= f'(a)\,h + f''(a)\,\frac{h^2}{2} + f'''(a+\theta h)\,\frac{h^3}{2\,.\,3}.$$

The process here used can obviously be carried to any number of terms; hence Theorems 1, 2, 3 are proved.

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